The Cauchy problem for the fast $p-$Laplacian evolution equation. Global Harnack principle and fine asymptotic behaviour

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Motivation: Asymptotic behavior of solutions to the Heat Equation

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \mathbb{R}^N, \ \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{HE}$$

with $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then, for any $1 \leq q \leq \infty$:

$$\|u(\cdot,t) - G(\cdot,t)\|_{L^q(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to \infty,$$

where $G(x,t)$ is the Gaussian kernel:

$$G(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

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Question: does this imply the Convergence in Relative Error (CRE)

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Answer: not always.
Difficulty: when $|x| \to \infty$, both u and G tend to 0.

Counterexample: Let $u_0(x) = G(x, C)$. Then $u(x,t) = G(x, t + C)$ solves the corresponding HE. Thus

$$\frac{u(x,t)}{G(x,t)} - 1 = \frac{G(x, t+C)}{G(x,t)} = \left(\frac{t}{t+C}\right)^{N/2} \exp\left(\frac{x^2}{4}\frac{C}{t(t+C)}\right) - 1$$

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Open problem: characterize the class of initial data u_0 for which (CRE) holds for the (HE).

Problem solved for nonlinear analogues

CRE holds when $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (not optimal) for the following equations:

- The fast diffusion equation $u_t(x,t) = \Delta u^m(x,t)$, $m_c = \frac{N}{N+2} < m < 1$.
  - Convergence in relative error: Vázquez (2003).
  - Global Harnack Principle: Bonforte-Vázquez (2006), under pointwise tail conditions on the data.
  - A complete characterization of the GHP and CRE recently proven by Bonforte and Simonov (2020), in the case of (FDE), also in presence of Caffarelli-Kohn-Nirenberg (CKN) weights.
- The fractional heat equation $u_t(x,t) = -(-\Delta)^s u(x,t)$, $0 < s < 1$.

  See Bonforte-Sire-Vázquez (2017).
- The $p-$Laplacian evolution equation $u_t(x,t) = \Delta_p u(x,t)$, $p_c = \frac{2N}{N+1} < p < 2$.

  See Bonforte-Simonov-Stan 2021.

The corresponding of the Gaussian kernel is the fundamental solution of each equation.

The evolution p-Laplacian equation

$$u_t(x,t) = \Delta_p u(x,t) \tag{PLE}$$

Goal: Prove quantitative lower and upper estimates for the Cauchy problem

$$\begin{cases} u_t(x,t) = \Delta_p u(x,t), & x \in \mathbb{R}^N, \ \ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{CP}$$

where $N \geq 1$, $p_c := \frac{2N}{N+1} < p < 2$ (good fast diffusion range), $u_0 \in L^1(\mathbb{R}^N)$

and $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian operator.

M. Bonforte, N. Simonov, D.Stan *The Cauchy problem for the fast p-Laplacian evolution equation. Characterization of the global Harnack principle and fine asymptotic behaviour*, preprint arXiv:2103.03312.

Fundamental solution for $p_c < p < 2$

The large time asymptotic behavior is described by the Barenblatt solution:

$$\mathcal{B}(x,t;M) = t^{\frac{1}{2-p}} \left[b_1 t^{\frac{\beta p}{p-1}} M^{\frac{p-2}{p-1}\beta p} + b_2 |x|^{\frac{p}{p-1}} \right]^{-\frac{p-1}{2-p}}$$

Figure: $B(x,t_1;M)$, $B(x,t_2;M)$, $B(x,t_3;M)$ for $t_3 < t_2 < t_1$.

Convergence of u to the fundamental solution with the same mass

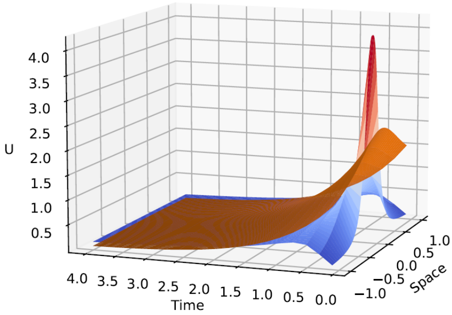

Figure: Orange: the solution to the (CP). Blue: the fundamental solution to the PLE with same mass

For $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$:

$$\|u(\cdot,t) - \mathcal{B}(\cdot,t;M)\|_{L^1(\mathbb{R}^N)} \to 0, \quad \text{as } t \to \infty,$$

$$t^{N\beta}\|u(\cdot,t) - \mathcal{B}(\cdot,t;M)\|_{L^\infty(\mathbb{R}^N)} \to 0, \quad \text{as } t \to \infty,$$

References: Kamin and Vázquez (1988) for $p > 2$. Bonforte-Simonov-Stan (2021) for $p_c < p < 2$.

The Cauchy problem: properties of solutions for $p_c < p < 2$

- Existence and uniqueness of weak solutions for L^1 data.
- Conservation of mass $\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx = M$, for all $t > 0$.
- Infinite speed of propagation (compactly supported initial data produce everywhere-positive solutions)
- *Benilan-Crandall estimate*: $u_t(\cdot,t) \leq \dfrac{u(\cdot,t)}{(2-p)t}$ for a.e. $t > 0$.

  $\Leftrightarrow$ for a.e. $x \in \mathbb{R}^N$ we have that

  $$t \to t^{-\frac{1}{2-p}} u(x,t) \quad \text{is a non-increasing function for a.e. } t > 0.$$
- $L^1 - L^\infty$ estimate: $\|u(t_0)\|_{L^\infty(\mathbb{R}^N)} \lesssim \|u_0\|_{L^1(\mathbb{R}^N)}^{p\beta} t_0^{-N\beta}$

Some references: DiBenedetto (1990), DiBenedetto-Gianazza-Vespri (book 2012), Vázquez (book 2006), Zhao (1995), ...

Other ranges of p

- $p = 2$: the linear *Heat Equation*, infinite speed of propagation and C^∞ smooth solutions obtained by the Gaussian representation formula.
- $p > 2$: degenerate or slow diffusion case, mass conservation, finite speed of propagation (compactly supported initial data generate solutions with compact support for all times), nonnegative integrable data give bounded solutions which are positive inside their support.
- $p \in (1, p_c)$: very fast diffusion regime, there is a regularity breakdown, mass is not preserved and solutions may extinguish in finite time. See Bonforte-Iagar-Vázquez (2010), DiBenedetto-Herrero (1989), DGV (book 2012).
- $p = 1$: very singular case $\Rightarrow$ the *Total Variation Flow* with applications in image processing. Even bounded solutions may be discontinuous. See Andreu-Caselles-Mazón (book 2004).

The questions we address

Do nonnegative integrable solutions behave like the fundamental solution?

If yes, in which sense? Do they have the same tail behaviour?

(Q)

Our space for initial data (Optimal for GHP and CRE)

Integral tail condition:

$$\mathcal{X}_p = \left\{ f \in L^1(\mathbb{R}^N) \ : \quad \sup_{R>0} R^{\frac{p}{2-p}-N} \int_{\mathbb{R}^N \setminus B_R(0)} |f(x)| dx \ < +\infty \right\}$$

Subspace: pointwise tail condition

$$\mathcal{A}_p = \left\{ f \in L^1(\mathbb{R}^N) \ : \exists A, R_0 > 0 \text{ s.t.} \quad |f(x)| \le A|x|^{-\frac{p}{2-p}} \quad \text{for all } |x| \ge R_0 \right\}$$

Then:

1. $C_0(\mathbb{R}^N) \subset \mathcal{A}_p \subset \mathcal{X}_p$.
2. $\mathcal{B}(x,t;M) \in \mathcal{A}_p \subset \mathcal{X}_p$.
3. $\mathcal{A}_p \subsetneq \mathcal{X}_p$.

Main result 1: GHP and CRE

Theorem

Let $N \geq 1$ and $p_c := \frac{2N}{N+1} < p < 2$. Let u be a weak solution to Problem (CP) corresponding $0 \leq u_0 \in L^1(\mathbb{R}^N)$. Then, the following statements are equivalent:

(i- Characterization in terms of the space $\mathcal{X}_p$)

$$u_0 \in \mathcal{X}_p \setminus \{0\} \qquad \text{that is} \qquad 0 < \sup_{R>0} R^{\frac{p}{2-p}-N} \int_{\mathbb{R}^N \setminus B_R(0)} |u_0(y)|dy < +\infty.$$

(ii- Global Harnack Principle). *For any $t_0 > 0$, there exist (explicit) constants τ_1, M_1, τ_2, M_2 such that, for all $x \in \mathbb{R}^N$ and $t > t_0$:*

$$\mathcal{B}(x, t - \tau_1; M_1) \leq u(x,t) \leq \mathcal{B}(x, t + \tau_2; M_2)\,.$$

(iii- Uniform Convergence in Relative Error) *We have that*

$$\lim_{t\to\infty} \left\| \frac{u(\cdot,t)}{\mathcal{B}(\cdot,t;M)} - 1 \right\|_{L^\infty(\mathbb{R}^N)} = 0\,, \quad \text{where} \quad M = \|u_0\|_{L^1(\mathbb{R}^N)}\,.$$

What happens when the initial datum is in $\mathcal{X}_p^{\mathbf{c}} = L^1(\mathbb{R}^N) \setminus \mathcal{X}_p$?

We construct explicit examples of sub-solutions and super-solutions to the (PLE), with a tail which is slightly fatter that the maximal one allowed in $\mathcal{X}_p$. As a consequence, we deduce that solutions corresponding to such data will never satisfy a GHP:

$$\frac{1}{(1+|x|)^{\frac{p}{2-p}-\delta}} \lesssim u_0(x) \lesssim \frac{1}{(1+|x|)^{\frac{p}{2-p}-\varepsilon}}$$

implies

$$\frac{c_0(t)}{(1+|x|)^{\frac{p}{2-p}-\delta}} \lesssim u(x,t) \lesssim \frac{c_1(t)}{(1+|x|)^{\frac{p}{2-p}-\varepsilon}}$$

for sufficiently small $\varepsilon, \delta > 0$, where c_0, c_1 are explicit functions that we construct.

Some applications

- Uniform convergence in relative error (equivalent to GHP by Theorem 1).
- Convergence rates.See for instance Agueh-Blanchet-Carrillo (2009), Del Pino-Dolbeault (2002), where (a stricter condition than) GHP is taken as an assumption to make all the machinery work. They work with the Doubly Nonlinear Diffusion equation $u_t = \Delta_p u^m$, so that when $m = 1$ you recover (PLE).
- Quantitative stability results in Gagliardo-Nirenberg-Sobolev inequalities, see Bonforte-Dolbeault-Nazaret-Simonov (2020).
- To describe the behavior of solutions to reaction-diffusion problems, see for instance Audrito-Vázquez (2017) for the doubly nonlinear reaction-diffusion equations and Stan-Vázquez (2014) for the fractional diffusion.

Local Harnack estimates

Local Harnack estimates for the (PLE): Bonforte-Iagar-Vázquez (2010)

$$\inf_{B_R(x_0)} u(x,t) \geq C \sup_{B_R(x_0)} u(x,t),$$

for all $R > 0$, with C independent of the ball $B_R(x_0)$.

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for all $R > 0$, with C independent of the ball $B_R(x_0)$.

- If you replace $B_R(x_0)$ by $\mathbb{R}^N$, the estimate does not say anything:

$$\inf_{\mathbb{R}^N} u(x,t) = 0.$$

- In $\mathbb{R}^N$ the equivalent result is the Global Harnack Principle: comparing u with the fundamental solution.

Towards the GHP: Upper bound

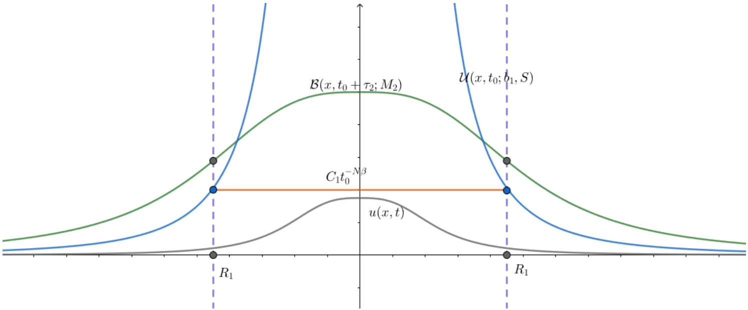
Theorem

Let $N \geq 1$ and $p_c < p < 2$. Let u be a weak solution to (CP) corresponding to an initial datum $0 \leq u_0 \in \mathcal{X}_p$. Then, for any $t_0 > 0$, there exist constants τ_2, M_2 such that

$$u(x,t) \leq \mathcal{B}(x, t+\tau_2; M_2), \qquad \textit{for all} \quad x \in \mathbb{R}^N,\ t \in (t_0, \infty).$$

Steps:

1. $u_0 \in \mathcal{X}_p \quad \Rightarrow \quad u(x, t_0/2) \leq A|x|^{-\frac{p}{2-p}}, \forall x \in \mathbb{R} \quad \Leftrightarrow u(\cdot, t_0/2) \in \mathcal{A}_p.$
2. Prove the upper bound for data in $\mathcal{A}_p$.
3. Relate the result to $u(x,t)$ using the semigroup structure of the (PLE).



Strategy:

1. Determine sufficient conditions s.t $u(x, t_0) \le \mathcal{U}(x, t; S)$ for $|x| > 0$, where $\mathcal{U}=$ singular Barenblatt solution.
2. $\mathcal{U}$ meets the upper bound given by the smoothing effect
$$\|u(t_0)\|_{L^\infty(\mathbb{R}^N)} \lesssim \|u_0\|_{L^1(\mathbb{R}^N)}^{p\beta} t_0^{-N\beta}$$
at some point $|x| = R_1$.
3. We find the Barenblatt solution $\mathcal{B}(x, t_0 + \tau_2; M_2)$ to be above the barrier $\mathcal{U}$ for all $|x| \ge R_1$, and therefore it will be above $u(x, t_0)$.
4. Inside the ball $\{|x| < R_1\}$ the comparison follows by the monotonicity of $\mathcal{B}(x, t_0 + \tau_2; M_2)$ in $|x|$.

Lower bound

Theorem

Let $N \geq 1$ *and* $p_c < p < 2$. *Let* u *be a weak solution to Problem (CP) with* $u_0 \in L^1(\mathbb{R}^N)$, *and let* $t_0 > 0$.

Then there exist $\underline{\tau} > 0$ *and* $\underline{M} > 0$ *such that*

$$u(x,t) \geq \mathcal{B}(x, t - \underline{\tau}; \underline{M}), \quad \text{for all } x \in \mathbb{R}^N \text{ and } \ t \geq t_0 \,,$$

where $\underline{\tau}$ *and* $\underline{M} > 0$ *are computed explicitly in terms of* t_0, M.

Approach to Lower Bound

Under the hypothesis of the previous theorem, we have:

$$\inf_{x\in B_R(x_0)} u(x,t) \geq \underline{C}\, \frac{M_R}{R^N} \cdot \begin{cases} \left(\frac{t}{t_c}\right)^{-N\beta} & \text{for } t \geq t_c, \\ \left(\frac{t}{t_c}\right)^{1/(2-p)} & \text{for } t \leq t_c, \end{cases}$$

where

$$t_c := \kappa\, M_R^{2-p}\, R^{\frac{1}{\beta}} \quad \text{and} \quad M_R := \int_{B_R(x_0)} u_0 \, dx \, .$$

Then, we find $B(x, t_c - \tau : \underline{M})$ such that

$$\inf_{x\in B_{R_0}(0)} u(x,t_c) \geq \sup_{x\in B_{R_0}(0)} \mathcal{B}(x, t_c - \underline{\tau}; \underline{M}) \, .$$

Outside the ball: use the Parabolic Comparison.

Convergence in relative error

Let $N \geq 1$ and $p_c < p < 2$. Let u be a weak solution to Problem (CP) corresponding $0 \leq u_0 \in \mathcal{X}_p \setminus \{0\}$ and let $M = \|u_0\|_{L^1(\mathbb{R}^N)}$. Then

$$\lim_{t\to\infty} \left\| \frac{u(\cdot,t)}{\mathcal{B}(\cdot,t;M)} - 1 \right\|_{L^\infty(\mathbb{R}^N)} = 0\,.$$

Ingredients:

1. Convergence to the Barenblatt profile in the L^∞ norm $\Rightarrow$ needed to control $|u - \mathcal{B}|$ in interior cones $\{|x| \leq Ct^\beta\}$ (balls in self-similar coordinates).
2. The GHP in the form of upper and lower bounds $\Rightarrow$ needed to control the relative error in exterior cones $\{|x| \geq Ct^\beta\}$.

Main result 2: Gradient Estimates

Theorem (Sharp Gradient Estimates)

Let $N \geq 1$ *and* $p_c < p < 2$. *Let* u *be the solution of Problem (CP) with* $0 \leq u_0 \in L^1(\mathbb{R}^N)$. *Then, there exists* $c_1 = c_1(N,p) > 0$ *s.t.*

$$\|\nabla u(t)\|_{L^\infty(\mathbb{R}^N)} \leq c_1 \frac{\|u_0\|_{L^1(\mathbb{R}^N)}^{2\beta}}{t^{(N+1)\beta}} \qquad \text{for any } t > 0.$$

Moreover, if $0 \leq u_0 \in \mathcal{X}_p$, *then there exists* $c_2 = c_2(N,p) > 0$ *s.t.*

$$|\nabla u(x,t)| \leq c_2 \frac{\|u_0\|_{L^1(\mathbb{R}^N)}^{2\beta} + \|u_0\|_{\mathcal{X}_p}^{2\beta} + t^{\frac{2\beta}{2-p}}}{(1+|x|)^{\frac{2}{2-p}}\, t^{(N+1)\beta}} \qquad \text{for any } x \in \mathbb{R}^N \text{ and } t > 0\,.$$

Sharpness:

1. $\|\nabla \mathcal{B}(\cdot,t,;M)\|_{L^\infty(\mathbb{R}^N)} = c\, t^{-(N+1)\beta}\, M^{2\beta}$, the maximum is taken on the curve $t^\beta = M^{(2-p)\beta}\, h\,|x|$ for $t > 0$, where $h = h(p,N) > 0$ is a constant.
2. For $t^\beta \leq C|x|$, ineq. (2) meets the space-time behaviour of the Barenblatt profile.
3. It is possible to construct counterexamples if $u_0 \notin \mathcal{X}_p$.

Based on a result of Zhao (1995).

Better result for radially decreasing initial data

Theorem

Let $N \geq 3$ and $p_c < p < 2$. Let u be the solution of Problem (CP) with datum $0 \leq u_0 \in \mathcal{X}_p \cap C^2(\mathbb{R}^N) \setminus \{0\}$ radial and nonincreasing, and let $M = \|u_0\|_{L^1(\mathbb{R}^N)}$. If there exist $A > 0$ and $R_0 > 0$ such that

$$|\partial_r u_0(r)| \leq A r^{-\frac{2}{2-p}}, \quad \text{for all } \ r \geq R_0 ,$$

then, the following limit holds

$$\left\| \frac{\partial_r u(\cdot, t)}{\partial_r \mathcal{B}(\cdot, t; M)} - 1 \right\|_{L^\infty(\mathbb{R}^N)} \xrightarrow{t \to +\infty} 0 ,$$

where $\partial_r u(r,t)$ (resp. $\partial_r \mathcal{B}(r,t;M)$) is the radial derivative of $u(r,t)$ (resp. $\mathcal{B}(r,t;M)$).

Equivalently: Radial data that decay faster (or equal) than the Barenblatt, and satisfy the corresponding tail condition for the radial derivative, produce solutions that converge uniformly in relative error to the Barenblatt with the same mass, in the C^1 topology.

Idea for the gradient decay proof in the radial data case

Fast diffusion equation:

$$\text{(FDE)} \qquad \overline{u}_t(\overline{x},t) = \Delta \overline{u}^m(\overline{x},t), \quad \overline{x} \in \mathbb{R}^N,\ t>0.$$

Radial version:

$$\text{(RFDE)} \Leftrightarrow \quad \overline{u}_t = \overline{r}^{1-\overline{n}} \frac{\partial}{\partial \overline{r}} \left(\overline{r}^{\overline{n}-1} \left| \overline{u} \right|^{m-1} \overline{u}_r \right), \quad \overline{x} \in \mathbb{R}^N.$$

Notice that:

- (RFDE) $\Leftrightarrow$ (FDE) for $\overline{n} = N$.
- For $\overline{n} > 0$: (RFDE) can be rewritten as Weighted Fast Diffusion Equation with Caffarelli-Kohn-Nirenberg weights

  $$\text{(WFDE)} \qquad \overline{u}_t = |x|^{\gamma}\, \nabla \cdot \left(|x|^{-\gamma} \left| \overline{u} \right|^{m-1} \nabla \overline{u} \right),$$

  where $\gamma = N - \overline{n}$.

See Iagar-Sanchez-Vázquez (2008), Bonforte-Dolbeault-Muratori-Nazaret (2017), Bonforte-Simonov (2020).

Connection between (PLE) and (RFDE)

PLE for radial solutions $u(r,t)$:

$$u_t = r^{1-N}\,\frac{\partial}{\partial r}\left(r^{N-1}\,|u_r|^{p-2}\,u_r\right), \quad x \in \mathbb{R}^N.$$

Theorem (Iagar-Sanchez-Vázquez (2008))

Suppose $2 < \overline{n} < \infty$. Then u and $\overline{u}$ are related through the following transformation:

$$\partial_r u(r,t) = D\,\overline{r}^{\frac{2}{m+1}}\,\overline{u}(\overline{r},t), \quad D = \left(\frac{(2m)^2}{m(m+1)^2}\right)^{\frac{1}{m-1}},$$

where

$$r = \overline{r}^{\frac{2m}{m+1}}, \quad p = m+1, \quad N = \frac{(\overline{n}-2)(m+1)}{2m}.$$

Thus, for radial solutions:

(PLE) $\Leftrightarrow$ (WFDE).

Next step: apply Bonforte-Simonov (2020) -Theorem 3.3 :

$$\left\| \frac{\overline{u}(\overline{r},t)}{-\mathfrak{B}(\overline{r},t;\overline{M})} - 1 \right\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to +\infty\,,$$

for $\overline{M}$ being the mass of the (negative) initial data

$$\overline{M} = -\int_{\mathbb{R}^N} \overline{u}_0(|x|)\,|x|^{-\gamma} dx$$

and $\mathfrak{B}(\overline{r},t;\overline{M})$ is the Barenblatt solution for the (WFDE).

Thank you!