Tutte characters for combinatorial coalgebras

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(joint work w/ Clément Dupont[2] and Luca Moci[3])

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"Narrow" deletion-contraction invariants

Definition

A graph or matroid function f is a narrow deletion-contraction invariant if

$$f(G) = f(G \setminus e) + f(G/e)$$

for any edge e of G other than a loop or an isthmus.

Theorem (Crapo)

A matroid function is a narrow d-c inv $\iff$ it is a specialisation of the Tutte polynomial

$$\mathfrak{T}(M; x, y) = \sum_{A \subset E(M)} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)} (y-1)^{|A|-\mathrm{rk}(A)}.$$

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Table 1. Settings with universal d-c invariants

Graph or matroid generalisation	**Universal invariant**
graphs; matroids	Tutte (1947), Crapo (1969)
polymatroids [Edmonds 1970]	Ardila–Aguiar (2017)
coloured matroids	Bollobás–Riordan (1999)
relative matroids [Brylawski 1972]. . .	Las Vergnas (1975)
graphs in surfaces; matroid perspectives [Las Vergnas 1975]	
ribbon graphs; delta-matroids [Bouchet 1987]	Bollobás–Riordan (2002)
graphs in pseudosurfaces	Krushkal (2011), Butler (2016)
arithmetic matroids [D'Adderio–Moci 2013]	Backman–Lenz (2016)

Prefactors

(Non)example

The graph chromatic polynomial $\chi(G)$ is not a narrow d-c inv. Instead

$$\chi(G) = \chi(G \setminus e) - \chi(G/e).$$

But

$$(-1)^{|E(G)|}\chi(G),$$

with a prefactor inserted, is a narrow d-c invariant.

Deletion-contraction invariants, take two

Definition

A graph or matroid function f is a deletion-contraction invariant if there exist functions N_1, N_2 of one-edge graphs s.t.

$$f(G) = N_1(G/e^c)f(G \setminus e) + N_2(G \setminus e^c)f(G/e)$$

for any edge e of G.

N_1 and N_2 also distinguish loops and isthmi from other edges.

Example (Tutte polynomial of matroids)

$\mathfrak{T}(M) = N_1(M/e^c)\,\mathfrak{T}(M \setminus e) + N_2(M \setminus e^c)\,\mathfrak{T}(M/e)$, where

$$N_1(e) = \begin{cases} y-1 & e \text{ a loop} \\ 1 & \text{else} \end{cases} \qquad N_2(e) = \begin{cases} x-1 & e \text{ a coloop} \\ 1 & \text{else.} \end{cases}$$

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Minors systems

Definition

A set species is a functor $\mathrm{Core}(\mathrm{FinSet}) \to \mathrm{Core}(\mathrm{FinSet})$, where $\mathrm{Core}(\mathrm{FinSet})$ is the category with finite sets as objects and bijections as morphisms.

$\mathcal{S}[E] = \{$objects with ground set $E\}$.

Definition (cf. Krajewski-Moffatt-Tanasa 2018, DHNKT 2013)

A minors system is a set species $\mathcal{S}[—]$ with

1. coproduct maps $\mathcal{S}[E] \to \mathcal{S}[A] \times \mathcal{S}[B]$ when $E = A \amalg B$
$$X \mapsto (X|A,\ X/A)$$
2. product maps $\oplus : \mathcal{S}[E] \times \mathcal{S}[\emptyset] \to \mathcal{S}[E]$

such that …

Notation. $X|A = X \setminus (E \setminus A)$.

Main theorems

$\mathcal{S}$: a minors system. R: a commutative ring. $f : \bigcup_E \mathcal{S}[E] \to R$.

Theorem (DFM)

If $$f(X) = N_1(X\backslash e^{\mathrm{c}})\, f(X/e) + N_2(X/e^{\mathrm{c}})\, f(X\backslash e),$$

where N_i are norms, and $f|_{\mathcal{S}[\emptyset]}$ is a twist, then $f = N_1 * f|_{\mathcal{S}[\emptyset]} * N_2$ is their convolution.

Theorem (DFM)

There is a universal d-c invariant $T_{\mathcal{S}}^{\mathrm{univ}} : \bigcup_E \mathcal{S}[E] \to (\ldots)$, the convolution of universal norms and a universal twist.

Metatheorem (DFM)

For $\mathcal{S}$ in Table 1, $T_{\mathcal{S}}^{\mathrm{univ}}$ is the universal invariant in the table "with variables duplicated to account for prefactors".

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Definitions from main theorem

Definition

$N : \bigcup_E \mathcal{S}[E] \to R$ is a norm if

1. N is constant on isomorphism classes,
2. $N(X) = N(X|A)N(X/A)$,
3. $N(Y) = 1$ for $Y \in \mathcal{S}[\emptyset]$.

Definition

$f : \mathcal{S}[\emptyset] \to R$ is a twist if it is multiplicative.

Definition

The convolution of $f, g : \bigcup_E \mathcal{S}[E] \to R$ is $f * g : \bigcup_E \mathcal{S}[E] \to R$,

$$(f * g)(X) = \sum_{A \subseteq E(X)} f(X|A)g(X/A).$$

Example: matroids

Proposition

$T^{\mathrm{univ}}_{\mathrm{Matroids}}$ *has codomain* $\mathbb{Z}[u_1, v_1, u_2, v_2]$.

$$T^{\mathrm{univ}}_{\mathrm{Matroids}}(M; u_1, v_1, u_2, v_2) = u_1^{\mathrm{rk}(M)}\, v_2^{|E(M)|-\mathrm{rk}(M)}\, \mathfrak{T}\left(M; \frac{u_2}{u_1}+1, \frac{v_1}{v_2}+1\right).$$

Conversely $\mathfrak{T}(M; x, y) = T^{\mathrm{univ}}_{\mathrm{Matroids}}(M; 1, y-1, x-1, 1)$.

Compare:

Example (Tutte polynomial of matroids)

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The universal norm on matroids is $N : \bigcup_E \text{Matroids}[E] \to \mathbb{Z}[u, v]$.

$$N(\text{coloop}) = u$$
$$N(\text{loop}) = v$$
$$N(M) = u^{\text{rk}(E(M))} v^{|E(M)|-\text{rk}(E(M))}$$

Matroids$[\emptyset] = \{\bullet\}$. The universal twist is Matroids$[\emptyset] \to \mathbb{Z}$, $\bullet \mapsto 1$.

Example: graphs

$T^{\text{univ}}_{\text{Graphs}}$ is different from $T^{\text{univ}}_{\text{Matroids}}$ due to connected components.

Example

Tutte's dichromate is a d-c invariant but not a matroid invariant:

$$Q(G; a, b) = \sum_{A \subset E} a^{h_0(V \cup A)}\, b^{|A| - |V| + h_0(V \cup A)}$$

Proposition

$T^{\text{univ}}_{\text{Graphs}}$ has codomain $\mathbb{Z}[u_1, v_1, a, u_2, v_2]$.

$$T^{\text{univ}}_{\text{Graphs}}(G) = \sum_{A \subseteq E(G)} u_1^{\mathrm{rk}(G|A)}\, v_1^{\mathrm{cork}(G|A)}\, a^{\#V(G|A/A)-1}\, u_2^{\mathrm{rk}(G/A)}\, v_2^{\mathrm{cork}(G/A)}$$

Example (cont.)

Example: graphs

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Example (cont.)

$$Q(G; a, b) = T^{\mathrm{univ}}_{\mathrm{Graphs}}(G; 1, b, a, 1, 1).$$

Example: ribbon graphs and delta-matroids

Proposition

$T^{\text{univ}}_{\Delta\text{Matroids}}$ maps to $\mathbb{Z}[u_1, v_1, w_1, u_2, v_2, w_2]/\langle w_1^2 - u_1 v_1, w_2^2 - u_2 v_2\rangle$.

$$T^{\text{univ}}_{\Delta\text{Matroids}}(D) = \sum_{A \subseteq E(D)} u_1^{\mathrm{rk}((D|A)_{\min})} v_1^{|A| - \mathrm{rk}((D|A)_{\max})} w_1^{\mathrm{rk}((D|A)_{\max}) - \mathrm{rk}((D|A)_{\min})} \cdot u_2^{\mathrm{rk}((D/A)_{\min})} v_2^{|E(D)| - \mathrm{rk}((D/A)_{\max})} w_2^{\mathrm{rk}((D/A)_{\max}) - \mathrm{rk}((D/A)_{\min})}.$$

Example (The bivariate Bollobás–Riordan "polynomial")

$$\widetilde{\mathfrak{R}}(D; x, y) = \sum_{A \subseteq E(D)} (x-1)^{\sigma(D) - \sigma(D|A)} (y-1)^{|A| - \sigma(D|A)}$$

equals $T^{\text{univ}}_{\Delta\text{Matroids}}(D; 1, y-1, x-1, 1, (x-1)^{\frac{1}{2}}, (y-1)^{\frac{1}{2}})$.

$\sigma(D) = \frac{1}{2}(\mathrm{rk}(D_{\max}) + \mathrm{rk}(D_{\min}))$.

Thanks!

Clément Dupont, Alex Fink and Luca Moci, Universal Tutte characters via combinatorial coalgebras, *Algebraic Combinatorics* **1** no. 5 (2018), 603–651. arXiv:1711.09028.