

# LAGRANGE MULTIPLIERS AND NONCONSTANT GRADIENT CONSTRAINED PROBLEM

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Aim of the talk is to study a gradient constrained problem associated with a linear operator.

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First, under a condition on the gradient constraint, we prove the equivalence of a nonconstant gradient constrained problem and a suitable obstacle problem, where the obstacle solves a Hamilton-Jacobi equation in the viscosity sense.

Second, we obtain the existence of Lagrange multipliers associated with the problem.

In particular, the existence of a Lagrange multiplier, which is a Radon measure, is established, whenever the free term  $f$  of the equation is of class  $L^p$ ,  $p > 1$ , whereas, if  $f$  is a positive constant, we regularize the result, namely we prove that it belongs to  $L^2$ .

These results are contained in



S. G., Lagrange multipliers and non-constant gradient constrained problem, J. Differential Equations 269 (2020), no. 1, 542-562.

# Outline

- Introduction

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- Results in literature

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- Conclusions

We consider  $\Omega \subset \mathbb{R}^n$ , an open bounded convex set with  $C^2$ -boundary  $\partial\Omega$ , and an operator

$$\mathcal{L}u = - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + cu$$

with associated bilinear form on  $H_0^{1,2}(\Omega) \times H_0^{1,2}(\Omega)$  given by

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} v + cuv \right) dx,$$

where

$$\left\{ \begin{array}{l} \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2 \text{ a.e. on } \Omega, \forall \xi \in \mathbb{R}^n \\ \nu > 0, a_{ij} \in C^2(\bar{\Omega}), b_i \in C^1(\bar{\Omega}) \\ c > 0 \text{ such large that } a(u, u) \geq \alpha \|u\|_{H_0^{1,2}(\Omega)}^2, \alpha > 0, \forall u \in H_0^{1,2}(\Omega). \end{array} \right.$$

(1)

We aim at studying the following non-constant gradient constrained problem, formulated by means of the variational inequality:

$$\text{Find } u \in K_{\nabla} = \left\{ v \in H_0^{1,2}(\Omega) : |Dv|^2 = \sum_{i=1}^n \left( \frac{\partial v}{\partial x_i} \right)^2 \leq g(x), \text{ a.e. on } \Omega \right\}$$

such that:

$$\int_{\Omega} \mathcal{L}u(v - u) dx \geq \int_{\Omega} f(v - u) dx, \quad \forall v \in K_{\nabla}, \quad (2)$$

with  $g(x) \in C^2(\overline{\Omega})$ ,  $g(x) > 0$ .

Moreover, we consider the obstacle problem

$$\text{Find } u \in K_w = \left\{ v \in H_0^{1,2}(\Omega) : |v(x)| \leq w(x) \text{ a.e. on } \Omega \right\}$$

such that

$$\int_{\Omega} \mathcal{L}u(v - u) dx \geq \int_{\Omega} f(v - u) dx, \quad \forall v \in K_w. \quad (3)$$

$w \in H^{1,\infty}(\Omega)$  is the viscosity solution of the Hamilton-Jacobi equation

$$\begin{cases} |Dw| = \sqrt{g(x)} & \text{a.e. in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \quad (4)$$

defined by P.L. Lions in



P.L. Lions, *Generalized Solutions for Hamilton-Jacobi Equations*, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1982

as

$$w(x) = \inf_{x_0 \in \partial\Omega} L(x, x_0) \quad (5)$$

$$L(x, x_0) = \inf \left\{ \int_0^{T_0} \sqrt{g(\xi(s))} ds : \xi : [0, T_0] \rightarrow \bar{\Omega}, \xi(0) = x, \xi(T_0) = x_0, \right. \\ \left. |\xi'(s)| \leq 1 \text{ a.e. in } [0, T_0] \right\} \quad (6)$$

For existence and uniqueness of solutions to (3) we refer to



J.-L. Lions, G. Stampacchia, Variational inequalities, Comm. pure and appl. Math. 20 (1967), p 493 - 51.

Obviously, in the case  $g(x) \equiv 1$ , the obstacle  $w(x)$  is the distance function.

The equivalence result is obtained under the following assumption on the gradient constraint  $g$

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial g}{\partial x_j} \right) \geq 0 \quad \text{in } \Omega. \quad (7)$$

Next, we provide a simple example, in which the solution to the obstacle problem (3) is not a solution to problem (2) and condition (7) is not satisfied.



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Next, we provide a simple example, in which the solution to the obstacle problem (3) is not a solution to problem (2) and condition (7) is not satisfied.

Also in the case  $\mathcal{L} = \Delta$  a condition on the sign of  $\Delta g$  is required, see



L. Santos, Variational problems with non-constant gradient constraints, *Port. Math.* 59 (2) (2002) 205-248.

The two problems are, in general, not equivalent.

The gradient constrained problem is a classical problem intensively studied a few decades ago.

In



L. Evans, A second order elliptic equation with gradient constraint, *Comm.Part. Diff. Eq.*, 4 (1979), 555-572

L.Evans studied general linear elliptic equations with a non-constant gradient constraint,  $g(x) \in C^2(\bar{\Omega})$ , and proved an existence and uniqueness result in the space  $W_{loc}^{2,p}(\Omega) \cap W_0^{1,\infty}(\Omega)$ , with  $1 < p < \infty$ .

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He also obtained  $W^{2,\infty}$  regularity under some additional restrictions. Those restrictions were removed by M. Wiegner in




M. Wiegner, The  $C^{1,1}$ -character of solutions of second order elliptic equations with gradient constraint, *Comm.Part. Diff. Eq.*, 6, 361-371 (1981).

Some extended results were obtained by H. Ishii and S. Koike, who prove that, if  $f$  and  $g$  are allowed to vanish simultaneously, uniqueness of solutions may fail.




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In

 R.Jensen, Regularity for elastoplastic type variational inequalities. *Indiana Univ. Math. J.* 32 (3), 407-423 (1983).

the author gives a  $W^{2,p}$ -regularity result for the solution of variational inequalities, which arise as minimization problems. 

In



G. Treu, M. Vornicescu, On the equivalence of two variational problems, *Calc. Var.* 11, 307-319 (2000)

the equivalence between a gradient constrained problem associated with an operator of type  $f(Du) + g(x, u)$  and a suitable obstacle problem is proved under some suitable regularity assumptions on  $g$  and the convexity condition on  $f$ .

The nonconstant gradient constraint is expressed by  $\gamma_{K^0}(Dv) \leq 1$ , where  $\gamma_K$  is the gauge function of  $K$  and  $K^0$  is the polar set of a compact convex subset of  $\mathbb{R}^n$ , whose interior contains the origin.

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Finally, in the recent paper



M. Safdari, Global optimal regularity for variational problems with nonsmooth non-strictly convex gradient constraints, *Journal of Differential Equations*, 279 (2021), 76-135

the author proves the optimal  $W^{2,\infty}$ -regularity for variational problems with convex gradient constraints.



In



L. Santos, Variational problems with non-constant gradient constraints, *Port. Math.* 59 (2) (2002) 205-248

the author obtains, under an extra condition on  $g$ , the equivalence between an evolutive variational inequality with non-constant gradient constraint associated with the Laplacian and a variational inequality with two obstacles and proves the existence of a Lagrange multiplier.

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The existence of Lagrange multipliers for non-constant gradient constraint problem associated with the Laplacian is also studied in



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Finally, in



A. Figalli, H. Shahgholian, An overview of unconstrained free boundary problems, *Philos. Trans. Roy. Soc. A* 373 (2015), no. 2050, 20140281.

the authors present an overview on obstacle problem and related problems, and provides here some references of applications.

## THEOREM

Under the above assumptions on  $\Omega$  and under assumption (1), if  $f \equiv \text{const.} > 0$ , the following conditions are satisfied

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial g}{\partial x_j} \right) \geq 0 \quad \text{in } \Omega$$

and

$$c \geq \|Db\|_{L^\infty} + \|D^2 a_{ij}\|_{L^\infty} + \frac{(3\|Da_{ij}\|_{L^\infty} + \|b\|_{L^\infty})^2}{4\nu}, \quad (8)$$

the solution  $u$  of

$$\text{Find } u \in K_\nabla = \left\{ v \in H_0^{1,2}(\Omega) : |Dv|^2 = \sum_{i=1}^n \left( \frac{\partial v}{\partial x_i} \right)^2 \leq g(x), \text{ a.e. on } \Omega \right\}$$

such that:

$$\int_{\Omega} \mathcal{L}u(v - u) dx \geq \int_{\Omega} f(v - u) dx, \quad \forall v \in K_\nabla,$$

## THEOREM

*coincides with the solution of*

Find  $u \in K_w = \left\{ v \in H_0^{1,2}(\Omega) : |v(x)| \leq w(x) \text{ a.e. on } \Omega \right\}$  such that:

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defined by

$$w(x) = \inf_{x_0 \in \partial\Omega} L(x, x_0)$$

$$L(x, x_0) = \inf \left\{ \int_0^{T_0} \sqrt{g(\xi(s))} ds : \xi : [0, T_0] \rightarrow \bar{\Omega}, \xi(0) = x, \xi(T_0) = x_0, |\xi'(s)| \leq 1 \text{ a.e. in } [0, T_0] \right\}$$

We would like to highlight that, in the proof of Theorem 1, we prove the following interesting coincidence of sets.

## THEOREM

*Under the same assumptions as in Theorem 1, let  $u \in K_{\nabla} \cap W^{2,p}(\Omega)$  be the solution to problem (2). Setting*

$$I = \{x \in \Omega : u(x) = w(x)\},$$

$$\Lambda = \{x \in \Omega : u(x) < w(x)\}$$

*it results*

$$P = \{x \in \Omega : |Du|^2 = g(x)\} = I,$$

$$E = \{x \in \Omega : |Du|^2 < g(x)\} = \Lambda.$$

Then, we are able to prove the following two results on the existence of Lagrange multipliers.

### THEOREM

*Under the same assumptions of Theorem 1, let  $u \in K_{\nabla} \cap W^{2,p}(\Omega)$  be the solution to problem (2). Then, there exists  $\bar{\mu} \in L^2(\Omega)$  such that*

$$\begin{cases} \bar{\mu} \geq 0 \text{ a.e. in } \Omega \\ \bar{\mu} \left( g(x) - \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i} \right)^2 \right) = 0 \text{ a.e. in } \Omega \\ \mathcal{L}u - f + \bar{\mu} = 0 \text{ a.e. in } \Omega. \end{cases} \quad (9)$$



## Remarks

It is easy to prove that, if  $u \in K_{\nabla}$  and there exists  $\bar{\mu}$  satisfying (9), then  $u$  is also the solution to problem (2).

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Let us also remark that condition (1) on the coefficients is verified in the case  $\mathcal{L} = \Delta$ .

Under condition (7) the regularity result by G. Cimatti holds, namely the solution  $u \in K_{\nabla}$  of (2) belongs to  $W^{2,p}(\Omega)$ .

## THEOREM

Under the above assumptions on  $\Omega$  and under assumption (1), let  $f \in L^p(\Omega)$ ,  $p > 1$ , and  $u \in K_{\nabla}$  be the solution to (2). Then, there exists  $\bar{\mu} \in (L^\infty(\Omega))^*$  such that

$$\left\{ \begin{array}{l} \langle \bar{\mu}, y \rangle \geq 0 \quad \forall y \in L^\infty(\Omega), y \geq 0 \quad \text{a.e. in } \Omega; \\ \langle \bar{\mu}, \left( \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i} \right)^2 - g(x) \right) \rangle = 0; \\ \int_{\Omega} (\mathcal{L}u - f)\varphi \, dx = \langle \bar{\mu}, -2 \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \rangle, \quad \forall \varphi \in H_0^{1,\infty}(\Omega) \end{array} \right. \quad (10)$$

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Since  $\bar{\mu} \in (L^\infty(\Omega))^*$ ,  $\bar{\mu}$  can be expressed by a Radon's integral with respect to the finitely additive measure  $\Psi$ :

$$\bar{\mu}(v) = \int_{\Omega} v(x) \Psi(dx).$$

Let  $\psi : S \rightarrow \mathbb{R}$ ,  $g : S \rightarrow Y$ , where  $S$  is a nonempty subset of a real linear space  $X$ ,  $Y$  is a partially ordered real normed space with the ordering cone  $C$ , and consider the primal problem:

$$\min_{\substack{g(v) \in -C \\ v \in S}} \psi(v) \quad (11)$$

The Lagrange dual problem is:

$$\max_{\mu \in C^*} \inf_{v \in S} [\psi(v) + \langle \mu, g(v) \rangle], \quad (12)$$

where  $C^* := \{\lambda \in Y^* : \langle \lambda, y \rangle \geq 0, \forall y \in C\}$  is the dual cone of  $C$ .

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As it is well known, it always results

$$\max_{\mu \in C^*} \inf_{v \in S} [\psi(v) + \mu(g(v))] \leq \min_{\substack{g(v) \in -C \\ v \in S}} \psi(v) \quad (13)$$

and, if problem (11) is solvable and in (13) the equality holds, we say that the strong duality between primal problem (11) and dual problem (12) holds.

## THEOREM (CLASSICAL STRONG DUALITY PROPERTY)

Let the composite mapping  $(\psi, g) : S \rightarrow \mathbb{R} \times Y$  be convex-like with respect to  $\mathbb{R}_+ \times C$  in  $\mathbb{R} \times Y$ . Let  $\mathbb{K} = \{v \in S : g(v) \in -C\} \neq \emptyset$  and let the ordering cone  $C$  have a nonempty interior  $\text{int}(C)$ . If the primal problem

$$\min_{v \in \mathbb{K}} \psi(v) \quad (14)$$

is solvable and the generalized Slater condition is satisfied, namely there is a vector  $\hat{v} \in S$  with  $g(\hat{v}) \in -\text{int}(C)$ , then the dual problem

$$\max_{\mu \in C^*} \inf_{v \in S} [\psi(v) + \mu(g(v))] \quad (15)$$

is also solvable and the extremal values of the two problems are equal. Moreover, if  $u$  is the optimal solution to problem (14) and  $\bar{\mu} \in C^*$  is a solution of the problem (15), it results

$$\bar{\mu}(g(u)) = 0. \quad (16)$$



## DEFINITION

The functional  $L : S \times C^* \rightarrow \mathbb{R}$  such that

$$L(v, \mu) = \psi(v) + \mu(g(v)) \quad \forall v \in S, \forall \mu \in C^*$$

is called *Lagrange functional*.

## THEOREM

*Under the same assumptions as above, suppose the ordering cone  $C$  to be closed. Then,  $u \in \mathbb{K}$  is an optimal solution to (11) if and only if there exists  $\bar{\mu} \in C^*$ , such that  $(u, \bar{\mu})$  is a saddle point of the Lagrange functional, namely:*

$$L(u, \mu) \leq L(u, \bar{\mu}) \leq L(v, \bar{\mu}), \quad \forall v \in S, \forall \mu \in C^*,$$

and

$$\langle \bar{\mu}, g(u) \rangle = 0.$$

However, the ordering cone of almost all the known problems, stated in infinite dimensional spaces, has the interior (and generalized interior concepts) empty.

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Hence, the above interior conditions cannot be used to guarantee the strong duality.

In order to overcome this serious drawback, which was also recognized by H. Brezis who writes *“Il semble toutefois que les techniques abstraites d’analyse fonctionnelle soient ici insuffisants”*,




P. Daniele, S. G., G. Idone and A. Maugeri in






P.Daniele P., S.G., G.Idone, A.Maugeri, Infinite Dimensional Duality and Applications. Math. Ann. 339, 221-239 (2007)

introduced a condition called *Assumption S* (from Separation), which ensures strong duality and has revealed itself very effective in applications.


This theory has been then refined in

-  P. Daniele, S. G., General infinite dimensional duality and applications to evolutionary network equilibrium problems. *Optim. Lett.* 1, 227-243 (2007);
-  P. Daniele, S.G., A. Maugeri, Remarks on General Infinite Dimensional Duality with Cone and Equality Constraints, *Commun. Appl. Anal.* 13 (2009) no.4 567–578;
-  P. Daniele, S. G., A. Maugeri, F.Raciti, Duality theory and applications to unilateral problems, *J. Optim. Theory Appl.*, 162 (3), 718-734 (2014).

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In

-  Grad A.: Quasi-Relative Interior-Type Constraints Qualifications Ensuring Strong Lagrange Duality For Optimization Problems with Cone and Affine Constraints. *J. Math. Anal. Appl.* 364, 86-95 (2010).

the author pointed out the surprising result that *Assumption S* is a necessary and sufficient condition.

We shall use the following main concepts and results.

Given a point  $x \in X$  and a subset  $M$  of  $X$ , the set

$$T_M(x) := \left\{ h \in X : h = \lim_n \lambda_n(x_n - x), \lambda_n > 0, x_n \in M \forall n \in \mathbb{N}, \lim_n x_n = x \right\}$$

is called the tangent cone to  $M$  at  $x$ .

If  $x \in clM$  (the closure of  $M$ ) and  $M$  is convex, we have

$$T_M(x) = clcone(M - \{x\}),$$

where the  $coneA = \{\lambda x : x \in A, \lambda \in \mathbb{R}^+\}$ .

Given  $\psi : S \rightarrow \mathbb{R}$ ,  $g : S \rightarrow Y$ ,  $h : S \rightarrow Z$  where  $S$  is a convex subset of a linear topological space  $X$ ,  $Y$  is a real normed space ordered by a convex cone  $C$ ,  $Z$  is a real normed space, consider the optimization problem

$$\min_{v \in \mathbb{K}} \psi(v), \quad (17)$$

where

$$\mathbb{K} = \{v \in S : g(v) \in -C, h(v) = \theta_Z\}$$

and  $\theta_Z$  is the zero element in the space  $Z$ .

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where

$$\mathbb{K} = \{v \in S : g(v) \in -C, h(v) = \theta_Z\}$$

and  $\theta_Z$  is the zero element in the space  $Z$ .

### DEFINITION (ASSUMPTION S)

*Assumption S* is fulfilled at a point  $u \in \mathbb{K}$  if and only if

$$T_{\tilde{M}}(0, \theta_Y, \theta_Z) \cap (\mathbb{R}^{--} \times \theta_Y \times \theta_Z) = \emptyset$$

where

$$\tilde{M} = \{(\psi(v) - \psi(u) + \alpha, g(v) + y, h(v)) : v \in S \setminus \mathbb{K}, \alpha \geq 0, y \in C\},$$

$$\mathbb{R}^{--} = \{\lambda \in \mathbb{R} : \lambda < 0\}.$$



## THEOREM (NEW STRONG DUALITY PROPERTY)

Assume that the functions  $\psi : S \rightarrow \mathbb{R}$ ,  $g : S \rightarrow Y$  are convex and that  $h : S \rightarrow Z$  is an affine-linear mapping. Assume that the Assumption  $S$  is fulfilled at the optimal solution  $u \in \mathbb{K}$  of the problem:  $\min_{v \in \mathbb{K}} \psi(v)$ .

Then also problem

$$\max_{\mu \in C^*, \lambda \in Z^*} \inf_{v \in S} [\psi(v) + \langle \mu, g(v) \rangle + \langle \lambda, h(v) \rangle], \quad (18)$$

where  $Z^*$  is the dual space of  $Z$  and  $C^*$  is the dual cone of  $C$ , is solvable. Moreover, if  $\bar{\mu} \in C^*$ ,  $\bar{\lambda} \in Z^*$  are optimal solutions to (18), we have

$$\langle \bar{\mu}, g(u) \rangle = 0 \quad (19)$$

and the optimal values of the two problems coincide; namely

$$\psi(u) = \min_{v \in \mathbb{K}} \psi(v) = \psi(u) + \langle \bar{\mu}, g(u) \rangle + \langle \bar{\lambda}, h(u) \rangle$$

The usual relationship with the saddle points of the Lagrange functional holds.

## THEOREM

*Let the assumptions of Theorem 10 be fulfilled. Then,  $u \in \mathbb{K}$  is an optimal solution to (11) if and only if there exist  $\bar{\mu} \in C^*$ ,  $\bar{\lambda} \in Z^*$  such that  $(u, \bar{\mu}, \bar{\lambda})$  is a saddle point of the Lagrange functional, namely:*

$$L(u, \mu, \lambda) \leq L(u, \bar{\mu}, \bar{\lambda}) \leq L(v, \bar{\mu}, \bar{\lambda}), \quad \forall v \in S, \forall \mu \in C^*, \forall \lambda \in Z^*$$

and

$$\langle \bar{\mu}, g(u) \rangle = 0.$$

In order to obtain the equivalence between Problems (2) and (3), the following proposition is proved.

### PROPOSITION

*Under the same assumptions of Theorem 1, the solution  $u$  of problem (3) satisfies*

$$|u|_1 = \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|} : x, y, \in \bar{\Omega}, x \neq y \right\} \leq |w|_1, \quad (20)$$

*with  $w(x)$  defined in (5).*

In the proof we argue in a similar way as done in



F.Rodrigues, *Obstacle Problems in Mathematical Physics*,  
North-Holland, Amsterdam, 1987,

in the case  $\mathcal{L}u = \Delta u$  and  $w(x) = \delta(x) = \text{dist}(x, \partial\Omega)$ .

We may easily prove that

$$K_{\nabla} \subset K_w. \quad (21)$$

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Then, let  $u \in K_w$  be the solution to (3).

Since  $f \equiv \text{const.} > 0$ ,  $\mathcal{L}$  is linear,  $u$  is also the solution of the problem

$$\text{Find } u \in K_1 : \int_{\Omega} \mathcal{L}u(v - u)dx \geq \int_{\Omega} f(v - u)dx, \quad \forall v \in K_1, \quad (22)$$

where

$$K_1 = \{v \in H_0^1(\Omega) : 0 \leq v(x) \leq w(x) \text{ a.e. on } \Omega\}.$$

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We may rewrite problem (22) as the optimization problem

$$\min_{v \in K_1} \psi(v) = \psi(u) = 0 \quad (23)$$

where  $\psi(v) = \int_{\Omega} (\mathcal{L}u - f)(v - u)dx$ .

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The optimization problem (23) fulfills *Assumption S* at  $u \in K_1 \cap W^{2,p}(\Omega)$  solution of (22). Then, strong duality holds.

## REMARK

We are considering  $Y = L^2(\Omega)$ , namely

$$C = \{w \in L^2(\Omega) : w(t) \geq 0 \text{ a.e. in } \Omega\}.$$

We cannot apply classical strong duality theory in the case, since the classical strong duality theory requires that  $\text{int}(C) \neq \emptyset$ .



Using variational arguments, we obtain that

$$\mathcal{L}u = f \quad \text{a.e. in } \Lambda = \{x \in \Omega : u(x) < w(x)\}. \quad (24)$$

Differentiating with respect to  $x_k$  and following the method used in



H. Brezis, G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques, Bull. Soc. Math. France 96 (1968) 153–180.

from the maximum principle, we obtain

$$|Du(x)| < \sqrt{g(x)} \quad \text{a.e. in } \Lambda$$

and then

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By uniqueness of the solutions of (3) and by (21), we may conclude that the solution to (3) is the solution to (2).

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By uniqueness of the solutions of (3) and by (21), we may conclude that the solution to (3) is the solution to (2).

In a similar way we obtain the existence a  $L^2$ - Lagrange multiplier.

Now, let us assume  $f \in L^p(\Omega)$ ,  $p > 1$ ,

We may rewrite problem (2) in the following way:

Find  $u \in K = \left\{ v \in H_0^{1,\infty}(\Omega) : \sum_{i=1}^n \left( \frac{\partial v}{\partial x_i} \right)^2 \leq g(x), \text{ a.e. on } \Omega \right\}$  such

that:

$$\int_{\Omega} \mathcal{L}u(v - u) dx \geq \int_{\Omega} f(v - u) dx, \quad \forall v \in K. \quad (25)$$

This will enable us to apply the classical duality theory in the case  $X = W^{1,\infty}(\Omega)$ ,  $Y = L^\infty(\Omega)$ , since in this case the ordering cone

$$C = \{w \in L^\infty(\Omega) : w(t) \geq 0 \text{ a.e. in } \Omega\}$$

has a nonempty interior.

Then, the existence of a Lagrange multiplier

$$\mu \in C^* = \{\mu \in (L^\infty(\Omega))^* : \mu(y) \geq 0 \forall y \in L^\infty(\Omega), y(x) \geq 0 \text{ a.a. } x \in \Omega\}$$

follows.

Problems (2) and (3) are, in general, not equivalent.

Indeed, we are going to present an example in which a solution to the obstacle problem (3) is not a solution to problem (2).

Let  $n = 1$ ,  $\Omega = (-1, 1)$ ,  $f \equiv 1$  and let  $\mathcal{L}$  be the operator given by

$$\mathcal{L}\varphi(x) = -\varphi''(x).$$

The function

$$u(x) = \frac{1}{2}(1 - x^2) \quad \text{for } x \in [-1, 1],$$

satisfies

$$\mathcal{L}u(x) = f(x) \quad \text{in } (-1, 1).$$

For  $\varepsilon > 0$ , we choose  $\theta_\varepsilon \in C^3(\mathbb{R})$  so that

$$\left\{ \begin{array}{ll} \theta_\varepsilon(r) = r & \text{for } r \leq \frac{1}{2}, \\ \theta'_\varepsilon(r) > 0, \theta''_\varepsilon(r) \leq 0 & \text{for } r \in \mathbb{R} \\ \theta'_\varepsilon(\frac{3}{4}) < \varepsilon & \\ \theta'''_\varepsilon(r) > 0 & r > \bar{r} > \frac{1}{2}. \end{array} \right.$$

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Setting

$$w_\varepsilon(x) = \theta_\varepsilon(h(x)) \quad \text{for } x \in [-1, 1],$$

where  $h(x) = 1 - |x|$ , it results  $w_\varepsilon \in W^{1,\infty}(\Omega)$  and

$$w'_\varepsilon(x) = \theta'_\varepsilon(h(x))h'(x) \quad \text{for } x \in (-1, 1) \setminus \{0\}.$$

Moreover, if  $w_\varepsilon(x) \leq \frac{1}{2}$ , then, by properties of  $\theta_\varepsilon$ ,

$$w_\varepsilon(x) = h(x) = 1 - |x|$$

and

$$u(x) = \frac{1}{2}(1 - x^2) \leq 1 - |x| = w_\varepsilon(x).$$

On the other hand, noting that

$$u(x) \leq \frac{1}{2} \quad \text{for } x \in (-1, 1),$$

we may conclude that

$$|u(x)| = u(x) \leq w_\varepsilon(x) \quad \text{for } x \in (-1, 1),$$

namely, with the choice of  $w = w_\varepsilon$ ,  $u$  is a solution to (3).



Next, if we set

$$g_\varepsilon(x) = (w'_\varepsilon(x))^2 \quad \text{for } x \in \Omega,$$

$w_\varepsilon$  is a viscosity solution to

$$|w'_\varepsilon(x)| = \sqrt{g_\varepsilon(x)} \quad \text{in } \Omega.$$

Selecting  $0 < \varepsilon < \frac{1}{4}$  and noting that

$$\left| w'_\varepsilon\left(\frac{1}{4}\right) \right| = \theta'_\varepsilon\left(\frac{3}{4}\right) < \varepsilon,$$

it results

$$\left| u'\left(\frac{1}{4}\right) \right| = \frac{1}{4} > \varepsilon > \left| w'_\varepsilon\left(\frac{1}{4}\right) \right| = \sqrt{g_\varepsilon\left(\frac{1}{4}\right)},$$

which shows that  $u \notin K_\nabla$  and, hence,  $u$  is not a solution to (2).

Let us stress that condition (7) is not verified. Indeed, if  $h(x) > \bar{r}$ , namely  $|x| < 1 - \bar{r}$

If the gradient constraint  $g$  is a positive constant, the problem reduces to the well-known elastic-plastic torsion problem.

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The formulation by R. Von Mises is the following one:

“The elastic-plastic torsion problem of a cylindrical bar with cross section  $\Omega$  is to find a function  $\psi(x)$  which vanishes on the boundary  $\partial\Omega$  and, together with its first derivatives, is continuous on  $\Omega$ ; nowhere on  $\Omega$  the gradient of  $\psi$  must have an absolute value (modulus) less than or equal to a given positive constant  $\tau$ ; whenever in  $\Omega$  the strict inequality holds, the function  $\psi$  must satisfy the differential equation




$$\Delta\psi = -2\mu\theta,$$

where the positive constants  $\mu$  and  $\theta$  denote the shearing modulus and the angle of twist per unit length respectively”.






R. Von Mises, *Mechanik der festen Körper im plastisch-deformablen Zustand*, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen Mathematisch-Physikalische Klasse* (1913) 582–592,


In the planar case the existence and the properties of a smooth solution of the problem have been studied by Ting, formulating it as a variational minimum problem

-  T. W. Ting, Elastic-plastic torsion of a square bar, *Transactions of the American Mathematical Society* **113** (1966) 369–401,
-  T. W. Ting, Elastic-Plastic Torsion Problem II, *Archive for Rational Mechanics and Analysis* **25** (1967) 342–366,
-  T. W. Ting, Elastic-plastic Torsion Problem III, *Archive for Rational Mechanics and Analysis* **34** (1969) 228–244.

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In multidimensional case in

-  H. Brezis, Multiplicateur de Lagrange en Torsion Elasto-Plastique, *Archive for Rational Mechanics and Analysis* **49** (1972) 32–40,

the author considers the variational inequality

Find  $u \in K = \left\{ v \in H_0^{1,\infty}(\Omega) : |Dv| = \sum_{i=1}^n \left( \frac{\partial v}{\partial x_i} \right)^2 \leq 1 \text{ a.e. on } \Omega \right\}$  such

that

$$\int_{\Omega} \sum_{i=1}^n \frac{\partial u}{\partial x_i} \left( \frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \geq \int_{\Omega} f(v - u) dx \quad \forall v \in K, \quad (26)$$

with  $f = \text{const.} > 0$ , and proves the existence of a Lagrange multiplier,

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with  $f = \text{const.} > 0$ , and proves the existence of a Lagrange multiplier, namely, if  $u$  is the solution of variational inequality (26), then there exists a unique  $\mu \in L^\infty(\Omega)$ ,  $\mu \geq 0$  a.e. in  $\Omega$  such that:

$$\begin{cases} \mu(1 - |Du|) = 0 \text{ a.e. in } \Omega \\ -\Delta u - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \mu \frac{\partial u}{\partial x_i} \right) = f \text{ in the sense of } D'(\Omega), \end{cases} \quad (27)$$

that is the solution of (26) solves the elastic-plastic torsion problem.

Conversely, if  $u \in K$  and there exists  $\mu$  satisfying (27), then  $u$  is the solution of (26).

In



S.G., A. Maugeri, New results on infinite dimensional duality in elastic-plastic torsion, *Filomat* 26 (5), 1029-1036 (2012;)



S.G., A. Maugeri, A measure-type Lagrange multiplier for the elastic-plastic torsion. *Nonlinear Analysis: Theory, Methods & Applications*, 102, 23-29 (2014);




P. Daniele, S. G., A. Maugeri, F. Raciti, Duality theory and applications to unilateral problems, *Journal of Optimization Theory and Applications*, 162 (3), 718-734 (2014).

we generalize the result by Brezis, proving that, in the case of linear operator  $\mathcal{L}$ ,

- the Lagrange multipliers associated with the elastic-plastic torsion problem always exist and, in general, they result as a Radon measure;
- under the assumption S,  $L^2$  Lagrange multipliers exist.



Moreover, in

 S. G., A. Maugeri, D. Puglisi, Elastic-Plastic Torsion Problem for Nonlinear Monotone Operators, J. Differential Equations 259 (2015), 817-837

we consider the nonlinear problem:

$$\text{Find } u \in K = \left\{ v \in W_0^{1,\infty}(\Omega) : |Dv| \leq 1 \text{ a.e. on } \Omega \right\}$$

such that

$$\int_{\Omega} \sum_{i=1}^n a_i(Du) \left( \frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) dx \geq \int_{\Omega} F(v - u) dx, \quad \forall v \in K.$$

and we prove that

- if the operator is strictly monotone, the Lagrange multipliers always exist and, in general, they result as a Radon measure;
- under strong monotonicity assumption, they are  $L^2$  functions.

## Future work

- The existence of  $L^2$ -Lagrange multipliers in the case of  $f$  regular.

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- The existence of  $L^2$ -Lagrange multipliers in the case of  $f$  regular.
- The existence of solutions with axial symmetry;
- The existence of Lagrange multipliers for the problem with non-constant gradient constraints associated with a nonlinear monotone operator.