Flattening knotted surfaces

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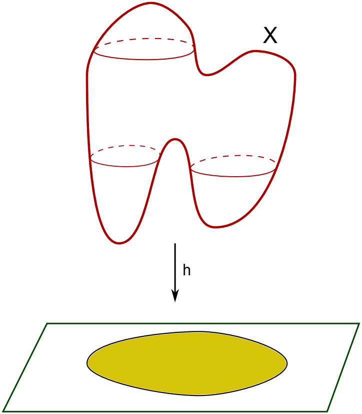
Overview

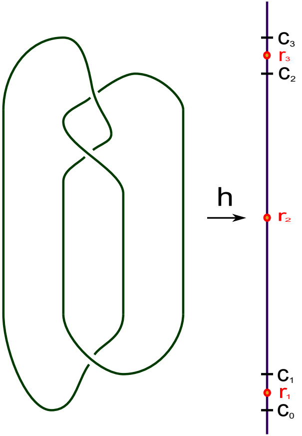
1. Introduction and motivation
2. Flattenings of surfaces
3. Invariants from flattenings

Fibrations of 4-manifolds

- Lefschetz fibrations
- broken Lefschetz fibrations
- Morse 2-functions
- trisections



Morse functions on classical knots



$\mathcal{M}(K) = \{h\colon S^3 \to \mathbb{R}$ with two critical points, restriction $h|_K$ is Morse$\}$

Invariants of classical knots, based on Morse functions

$\mathcal{M}(K) = \{h \colon S^3 \to \mathbb{R} \text{ with two critical points, restriction } h|_K \text{ is Morse}\}$

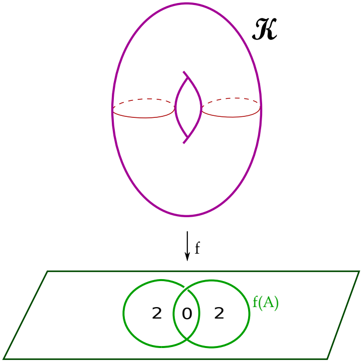
$$b(h) = \#\text{ of maxima of } h|_K \ , \quad \operatorname{trunk}(h) = \max_{1 \leq i \leq n} |K \cap h^{-1}(r_i)| \ ,$$

$$w(h) = \sum_{i=1}^{n} |K \cap h^{-1}(r_i)|$$

- bridge number: $b(K) = \min_{h \in \mathcal{M}(K)} b(h)$
- trunk of a knot: $\operatorname{trunk}(K) = \min_{h \in \mathcal{M}(K)} \operatorname{trunk}(h)$
- knot width: $w(K) = \min_{h \in \mathcal{M}(K)} w(h)$

Proposition

Let $\mathcal{K}$ be an embedded surface in a smooth compact 4-manifold X (the embedding being proper at the boundary if necessary), and let Σ be a 2-manifold. Consider a smooth map $f : X \to \Sigma$ and denote by $A \subset \mathcal{K}$ the set of critical points of $f|_{\mathcal{K}}$. Then the cardinality $|\mathcal{K} \cap f^{-1}(x)|$ is constant on each connected component of $\Sigma \backslash f(A)$.



Hyperbolic splitting of a surface

For a Morse function $h\colon S^4 \to \mathbb{R}$, we denote $S^4_t = h^{-1}(t)$ and $\mathcal{K}_t = \mathcal{K} \cap h^{-1}(t)$.

Definition

A Morse function $h\colon S^4 \to \mathbb{R}$ is called a **hyperbolic splitting** of an embedded surface $\mathcal{K}$ if it satisfies the following conditions:

1. h has exactly two critical points on S^4,
2. $h_{\mathcal{K}}$ is also Morse,
3. all minima of $h_{\mathcal{K}}$ occur in the level S^4_{-1},
4. all maxima of $h_{\mathcal{K}}$ occur in the level S^4_1,
5. all hyperbolic points of $h_{\mathcal{K}}$ occur in the level S^4_0.

Marked graph diagram

- h a hyperbolic splitting of a knotted surface $\mathcal{K}$
- the 0-section $\mathcal{K}_0$ is an embedded 4-valent graph
- vertices correspond to saddles
- $p\colon S^4_0 \to \Sigma$ takes $\mathcal{K}_0$ to its diagram $\Gamma = p(\mathcal{K}_0)$
- a marker at a vertex determines the resolutions

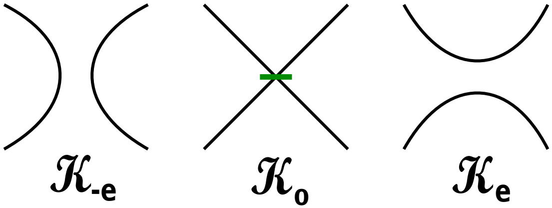
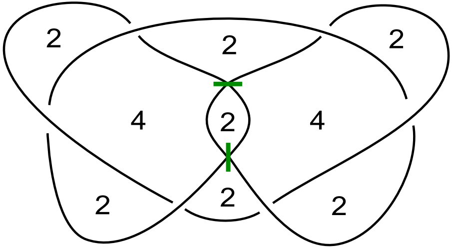


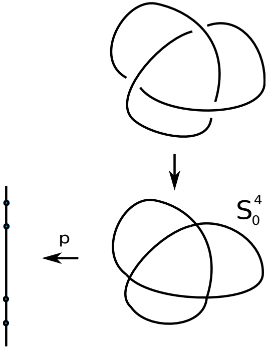
Figure: The resolutions of a vertex, corresponding to a marker

A flattening of a knotted surface



- $\mathcal{K}$ a knotted surface in S^4
- $h\colon S^4 \to \mathbb{R}$ a hyperbolic splitting of $\mathcal{K}$, $c(h)$ the critical points of h
- h defines a marked graph diagram Γ of $\mathcal{K}$
- h induces a map $h^\perp\colon S^4 \backslash c(h) \to \Sigma$
- the set of critical values of $h^\perp_{\mathcal{K}}$ is equivalent to Γ

The definition of a flattening



- $h\colon S^4 \to \mathbb{R}$ a hyperbolic decomposition of $\mathcal{K}$
- $\Phi_t(x)$ the flow of the vector field $\mathrm{grad}(h)$
- $p\colon S^4_0 \to \Sigma$ a regular projection to a 2-sphere Σ
- $h^{\perp}\colon S^4 \backslash c(h) \to \Sigma$ is defined by $h^{\perp}(x) = p(\Phi_t(x) \cap S^4_0)$

Properties of the flattening map

- $h^{\perp}\colon S^4 \backslash c(h) \to \Sigma$ is defined by $h^{\perp}(x) = p(\Phi_t(x) \cap S^4_0)$

Lemma

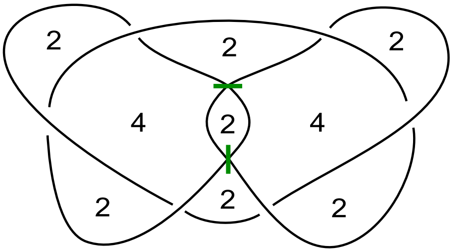
The map $h^{\perp}\colon S^4 \backslash c(h) \to \Sigma$ is smooth.

The restriction $h^{\perp}_{\mathcal{K}}\colon \mathcal{K} \to \Sigma$ is called a **flattening map**.

Proposition

Let h be a hyperbolic splitting of an embedded surface $\mathcal{K}$ with a marked graph diagram $\Gamma = p(\mathcal{K}_0)$. The set of critical values of the flattening map $h^{\perp}_{\mathcal{K}}\colon \mathcal{K} \to \Sigma$ equals Γ.

Fibers of the flattening map

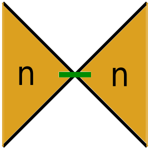


- flattening map $h^{\perp}_{\mathcal{K}} \colon \mathcal{K} \to \Sigma$
- the set of critical points of $h^{\perp}_{\mathcal{K}}$ is the diagram Γ
- components of $\Sigma \backslash \Gamma$ are called **regions**
- **multiplicity** in a region U is the cardinality of the fibers above U:

$$m_{h^{\perp}_{\mathcal{K}}}(U) = |\mathcal{K} \cap (h^{\perp})^{-1}(x)| \text{ for any } x \in U$$

Construction of invariants

- $\mathcal{H}(\mathcal{K}) = \{\text{hyperbolic splittings of } \mathcal{K}\}$
- each $h \in \mathcal{H}(\mathcal{K})$ defines a flattening map $h_{\mathcal{K}}^{\perp} \colon \mathcal{K} \to \Sigma$
- the set of critical points of $h_{\mathcal{K}}^{\perp}$ is the diagram Γ
- a vertex of Γ is called **inessential** if it is a marked vertex that represents a branch point of the flattening map
- equivalence relation on the set of regions of Γ



Invariants from flattenings

- $U_0, U_1, \ldots, U_n$ the regions of Γ
- equivalence relation $\sim$ on the index set $\{0, 1, \ldots, n\}$ by $i \sim j \Leftrightarrow U_i \text{ is equivalent to } U_j$.
- the quotient set $\mathcal{I} = \{[i] \mid i \in \{0, 1, \ldots, n\}\}$

$$p(h) = \#\left\{[i] \in \mathcal{I} \mid m_{h_{\mathcal{K}}^{\perp}}(U_i) > 0\right\} \ ,$$

$$\mathrm{trunk}\,(h) = \max_{[i]\in\mathcal{I}} m_{h_{\mathcal{K}}^{\perp}}(U_i) \ ,$$

$$w(h) = \sum_{[i]\in\mathcal{I}} m_{h_{\mathcal{K}}^{\perp}}(U_i)$$

Invariants from flattenings

$$p(\mathcal{K}) = \min_{h \in \mathcal{H}(\mathcal{K})} p(h) \quad \text{the partition number}$$

$$\mathrm{trunk}\,(\mathcal{K}) = \min_{h \in \mathcal{H}(\mathcal{K})} \mathrm{trunk}\,(h) \quad \text{the trunk of } \mathcal{K}$$

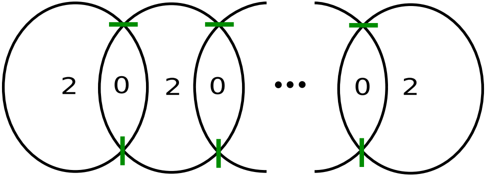
$$w(\mathcal{K}) = \min_{h \in \mathcal{H}(\mathcal{K})} w(h) \quad \text{the width of } \mathcal{K}$$

Proposition

For any smoothly embedded closed surface $\mathcal{K}$ we have

$$w(\mathcal{K}) \geq 2p(\mathcal{K}) + \mathit{trunk}\,(\mathcal{K}) - 2\ .$$

Basic results



Lemma

Let $\mathcal{K}$ be an orientable closed surface in S^4. If $\mathcal{K}$ is unknotted, then $p(\mathcal{K}) = 1$ and $w(\mathcal{K}) = \mathit{trunk}(\mathcal{K}) = 2$.

Connected sum of surfaces

Proposition

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed connected embedded surfaces in S^4, then

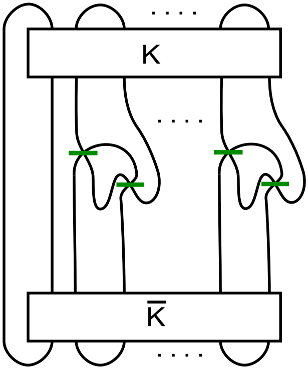
$$p(\mathcal{K}_1 \# \mathcal{K}_2) \leq p(\mathcal{K}_1) + p(\mathcal{K}_2) - 1 \, , \quad w(\mathcal{K}_1 \# \mathcal{K}_2) \leq w(\mathcal{K}_1) + w(\mathcal{K}_2) - 2 \, ,$$
$$and \;\; trunk(\mathcal{K}_1 \# \mathcal{K}_2) \leq \max\{trunk(\mathcal{K}_1), trunk(\mathcal{K}_2)\} \, .$$

Spun knots

Theorem

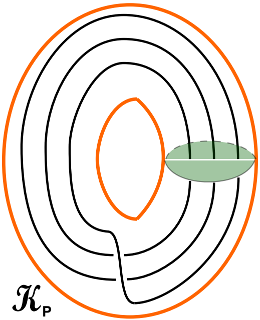
Let K be a 1-knot with bridge number $b(K)$. Denote by $\mathcal{S}(K)$ the spin of K. Then

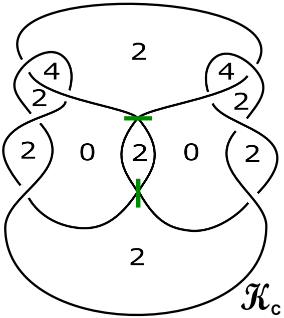
$$trunk(\mathcal{S}(K)) \leq 2b(K)\ .$$



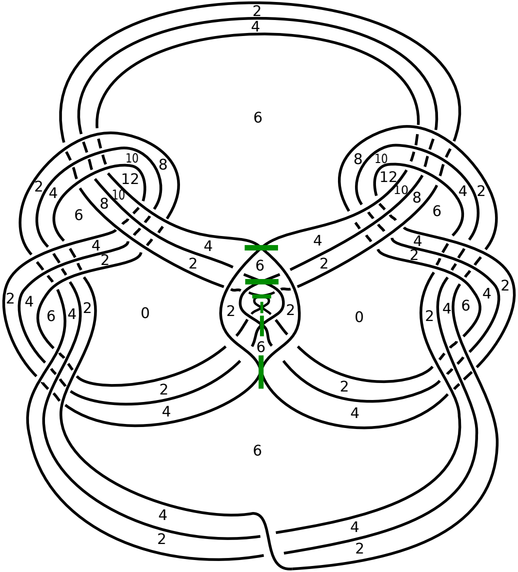
Satellite 2-knots

- $\mathcal{K}_P$ a 2-sphere embedded in $S^2 \times D^2$ (a pattern)
- $\mathcal{K}_C$ a 2-sphere embedded in S^4 with a tubular neighborhood $\nu(\mathcal{K}_C)$
- $f: S^2 \times D^2 \to \nu(\mathcal{K}_C)$ a diffeomorphism
- $f(\mathcal{K}_P)$ is a **satellite knot** with **pattern** $\mathcal{K}_P$ and **companion** $\mathcal{K}_C$.





Flattening of a satellite knot



Invariants of satellite knots

Theorem

Let $\mathcal{K}$ be a satellite 2-knot with companion $\mathcal{K}_C$ and pattern $\mathcal{K}_P$. Denote by ω the geometric winding of $\mathcal{K}_P$. Then

$$trunk(\mathcal{K}) \leq \max\{\omega\, trunk(\mathcal{K}_C), trunk(\mathcal{K}_P)\} \ .$$

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Thank you :)