On hysteresis reaction-diffusion systems and an application in population dynamics

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Outline

1. Singular fast-reaction limit derivation of hysteresis RD systems
2. Nonlinear hysteresis feedback and hysteresis induced blow up
3. Weak differentiability of a class of control-to-state maps

Outline

A population-stock model

We consider a (nonlinear) population dynamical PDE-ODE model

$$\begin{aligned} \partial_t u - D\Delta u &= \lambda(N, F, S)\, u && \text{in } [0, T] \times \Omega, \\ \partial_\nu u &= 0 && \text{on } [0, T] \times \partial\Omega, \\ u(0) &= u_{in} && \text{in } \Omega, \end{aligned}$$

with total population $N = \int_\Omega u(x)dx \;\Rightarrow\; \dot{N}(t) = \lambda N(t),\; N_{in} = \int_\Omega u_{in} dx$.

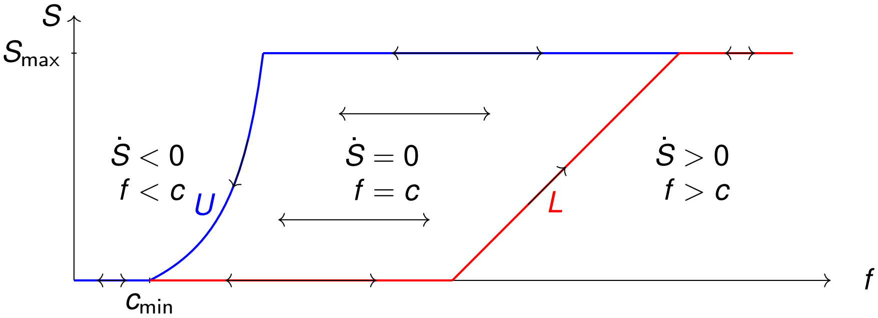
A given F food supply feeds a stock S with fast ($\varepsilon \ll 1$) turnover

$$\begin{aligned} \varepsilon \dot{S} &= N\,(f - c(N, f, S)) && \text{in } [0, T], \\ S(0) &= S_{in} \geq 0, \end{aligned}$$

$f = F/N$ individual food supply,

The $c(N, f, S)$ food consumption rate depends on N, f and S.

Phase space diagram of the stock dynamics



An unbounded consumption rate ensuring limited stock $S \leq S_{\max}$

$$c = \begin{cases} f + \frac{S}{N}\left(1 - e^{-N\left(1 - f/U^{-1}(S)\right)}\right) & \text{if } f < U^{-1}(S), \\ f & \text{if } U^{-1}(S) \leq f \leq L^{-1}(S), \\ fe^{-(S_{\max}-S)_+} + L^{-1}(S)\left(1 - e^{-(S_{\max}-S)_+}\right) & \text{if } f > L^{-1}(S). \end{cases}$$

Main theorem: singular limit to hysteresis RD system

Theorem Under "natural" assumptions holds in the limit $\varepsilon \to 0$

$$u_\varepsilon \to u_0 \quad \text{in} \quad \mathrm{W}^{1,q}(0,T;\mathrm{L}^2(\Omega)) \cap \mathrm{L}^q(0,T;\mathrm{H}^2(\Omega)) \qquad \forall q \in [2,\infty)$$
$$S_\varepsilon \to S_0 \quad \text{in} \quad \mathrm{L}^q(0,T).$$

Limit (u_0, S_0) solves uniquely the hysteresis (general. play) RD system

$$\partial_t u_0 - D\Delta u_0 = \left(\frac{c(S_0, N_0, F)}{c_{\min}} - 1\right) u_0 \qquad \text{a.e. in } (0,T)\times\Omega,$$
$$\partial_\nu u_0 = 0 \qquad \text{a.e. in } (0,T)\times\partial\Omega,$$
$$u_0(0) = u_{in} \qquad \text{a.e. in } \Omega,$$
$$\dot{S}_0(t)(S_0(t) - z) \leq 0 \quad \text{for all} \quad z \in [L(f_0(t)), U(f_0(t))] \qquad \text{a.e. in } [0,T],$$
$$S_0(0) = \min\{\max\{L(f_0(0)), S_{in}\}, U, (f_0(0))\} \, f_0(0) = F(0)/N_0(0),$$
$$S_0(t) \in [L(f_0(t)), U(f_0(t))] \qquad \text{in } [0,T].$$

Main theorem: singular limit to hysteresis RD system

Proof of Theorem One key problem is lack uniform bounds on $\dot{S}_\varepsilon$. This is bypassed by introducing a projection operator p_ε and suitable error/monotonicity estimates.

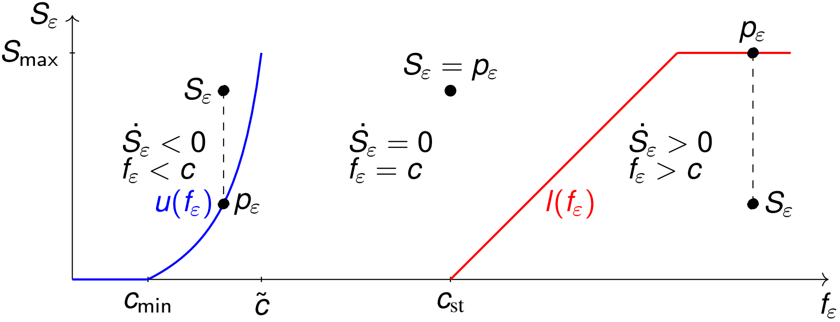


Figure: Sign of the gradient of S_ε and projection to p_ε in the f_ε-S_ε-phase diagram.

A numerical example with periodic food supply

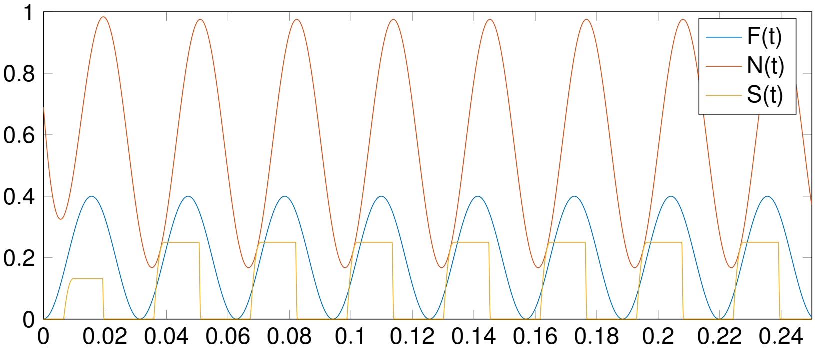


Figure: Evolution of the population-hysteresis-diffusion system subject to the time-periodic food supply $F(t) = 0.2(1 - \cos(t))$ (blue) and the resulting population size N (red) and stock S (yellow).

Outline

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A simplest hysteresis reaction-diffusion model

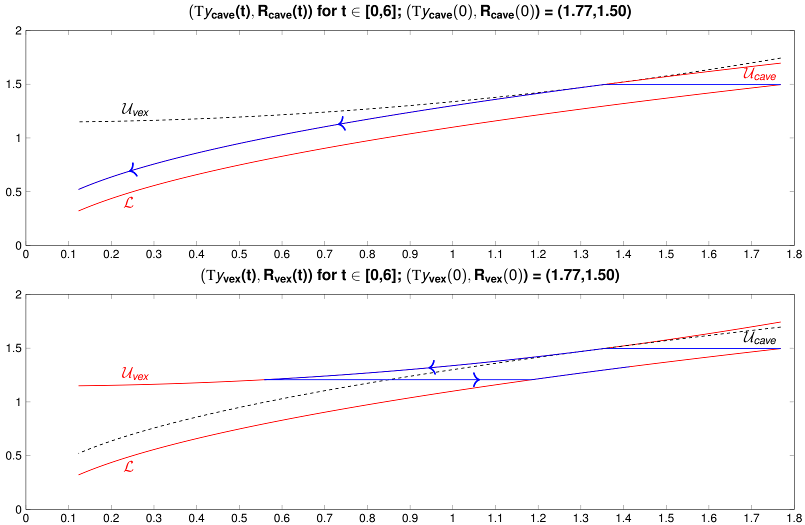
We consider

$$\begin{aligned} &\partial_t y - D\Delta y = Ry && \text{on} \quad \Omega \times (0,\infty), \\ &\partial_\nu y = 0 && \text{on} \quad \partial\Omega \times (0,\infty), \\ &y(0) = y_0 && \text{on} \quad \Omega, \end{aligned}$$

$Ry = R(\mathrm{T}y, R_0)$ is a generalised scalar play operator and $\mathrm{T}y$ is a projection onto spatially heterogeneous eigenfunctions $\{\phi_k\}_{k\geq 1}$

$$\mathrm{T}y := \underbrace{k_m}_{>0} \langle y, \phi_m\rangle + \sum_{i=m+1}^{M-1} \underbrace{k_i}_{\geq 0} \langle y, \phi_i\rangle + \underbrace{k_M}_{>0} \langle y, \phi_M\rangle, \qquad 1 \leq m < M$$

Spatial homogenisation versus grow-up



Outline

Model

We consider the control problem

$$y_t - \Delta y = u + \mathcal{W}[y], \qquad \text{in} \quad \Omega_T, \tag{1a}$$

$$\mathcal{B}[y] = 0, \qquad \text{on} \quad \Gamma_T, \tag{1b}$$

$$y(\cdot, 0) = y_0, \qquad \text{on} \quad \Omega. \tag{1c}$$

$\mathcal{B}$ is a standard mixed Dirichlet Neumann boundary operators.

$\mathcal{W}$ is a space-dependent version of a scalar operator $\mathcal{V}$, i.e.

$$\mathcal{W}[y](x,t) = \mathcal{V}[y(x,\cdot)](t), \qquad (x,t) \in \Omega \times [0,T].$$

Thus, $\mathcal{W}$ represents a family of operators acting on $y(x,\cdot)$, viewed as a function of time, at every $x \in \Omega$.

Underlying assumptions

- Lipschitz continuous Volterra operator, i.e. there exists an $L > 0$:

$$\mathcal{V} : \begin{cases} C[0,T] \to C[0,T], \\ |\mathcal{V}[v](t) - \mathcal{V}[\tilde{v}](t)| \leq L \sup_{0 \leq s \leq t} |v(s) - \tilde{v}(s)|, \end{cases}$$

  for every $v, \tilde{v} \in C[0,T]$ and every $t \in [0,T]$.

- Linear growth

$$|\mathcal{V}[v](t)| \leq L \sup_{0 \leq s \leq t} |v(s)| + c_0$$

  for the same arguments as above and some $c_0 > 0$.

Satisfied by many hysteresis operators, see [BS,Vis,MR].

Existence of the control-to-state map

[Visintin 1994]: System (1) has a unique solution and a well-defined control-to-state operator for any given $u \in L^2(\Omega_T)$

$$y = Su \,, \qquad S : L^2(\Omega_T) \to H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \,,$$

If S were Fréchet differentiable,
we would have for an increment $h \in L^2(\Omega_T)$

$$S(u+h) = Su + S'(u)h + o(\|h\|) \,,$$

and $d = S'(u)h$ would be the linear first order approximation to the difference $S(u+h) - Su$.

First order system

For given $y = Su$ and h, determine functions d and ω as solutions of

$$
\begin{aligned}
d_t - \Delta d &= h + \omega, && \text{in} \quad \Omega_T, && \text{(2a)}\\
\omega &= \mathcal{W}'[y; d], && \text{in} \quad \Omega_T, && \text{(2b)}\\
\mathcal{B}[d] &= 0, && \text{on} \quad \Gamma_T, && \text{(2c)}\\
d(\cdot, 0) &= 0, && \text{on} \quad \Omega. && \text{(2d)}
\end{aligned}
$$

Here, $\omega = \mathcal{W}'[y; d]$ stands for some type of derivative of $\mathcal{W}$ at y which involves the direction d.
The first order system is nonlinear if the mapping $d \mapsto \omega$ is nonlinear.

We do not assume that the derivative depends linearly on the direction d, which is not true for hysteresis operators.

Main result: Bouligand and Newton differentiability

[Brokate]: $\mathcal{V}$ is Bouligand and Newton differentiable as operator from $W^{1,p}(0,T)$ to $L^r(0,T)$ for $1 < p < \infty$.

If $F : O \subset X \to Y$ possesses a directional derivative $F^{BD}(u;h)$ for all $u \in O$, $h \in X$ with the property that

$$\lim_{h \to 0} \frac{\|F[u+h] - F[u] - F^{BD}[u;h]\|}{\|h\|} = 0,$$

then F is **Bouligand differentiable** with Bouligand derivative F^{BD}.

Theorem *The control-to-state mapping $u \mapsto y = Su$ is Bouligand resp. Newton differentiable when considered as an operator*

$$S : L^{2+\epsilon}(0,T;L^\infty(\Omega)) \to H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;V)$$

for sufficiently small $\epsilon > 0$. Moreover, the derivative is given by the solution d of the first order problem (2).

Summary

Summary:

- Hysteresis RD model as fast reaction limits
- Hysteresis diffusion driven instability
- Hysteresis RD models open for optimality conditions and semi-smooth Newton methods
- Proofs need to deal with the hysteresis non-locality in time

THANK YOU!