On hysteresis reaction-diffusion systems and an application in population dynamics

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1. Singular fast-reaction limit derivation of hysteresis RD systems

2. Nonlinear hysteresis feedback and hysteresis induced blow up

3. Weak differentiability of a class of control-to-state maps
Outline

1. Singular fast-reaction limit derivation of hysteresis RD systems
2. Nonlinear hysteresis feedback and hysteresis induced blow up
3. Weak differentiability of a class of control-to-state maps
A population-stock model

We consider a (nonlinear) population dynamical PDE-ODE model

\[ \partial_t u - D \Delta u = \lambda(N, F, S) u \quad \text{in } [0, T] \times \Omega, \]
\[ \partial_\nu u = 0 \quad \text{on } [0, T] \times \partial \Omega, \]
\[ u(0) = u_{in} \quad \text{in } \Omega, \]

with total population \( N = \int_\Omega u(x) dx \Rightarrow \dot{N}(t) = \lambda N(t), \ N_{in} = \int_\Omega u_{in} dx. \)

A given \( F \) food supply feeds a stock \( S \) with fast (\( \varepsilon \ll 1 \)) turnover

\[ \varepsilon \dot{S} = N \left( f - c(N, f, S) \right) \quad \text{in } [0, T], \]
\[ S(0) = S_{in} \geq 0, \]

\( f = F/N \) individual food supply,

The \( c(N, f, S) \) food consumption rate depends on \( N, f \) and \( S \).
Phase space diagram of the stock dynamics

An unbounded consumption rate ensuring limited stock $S \leq S_{\text{max}}$

\[
c = \begin{cases} 
  f + \frac{S}{N} \left(1 - e^{-N\left(1 - f/U^{-1}(S)\right)}\right) & \text{if } f < U^{-1}(S), \\
  f & \text{if } U^{-1}(S) \leq f \leq L^{-1}(S), \\
  fe^{-(S_{\text{max}} - S)_+} + L^{-1}(S) \left(1 - e^{-(S_{\text{max}} - S)_+}\right) & \text{if } f > L^{-1}(S). 
\end{cases}
\]
Main theorem: singular limit to hysteresis RD system

**Theorem** Under "natural" assumptions holds in the limit $\varepsilon \to 0$

\[ u_\varepsilon \to u_0 \quad \text{in} \quad W^{1,q}(0, T; L^2(\Omega)) \cap L^q(0, T; H^2(\Omega)) \quad \forall q \in [2, \infty) \]
\[ S_\varepsilon \to S_0 \quad \text{in} \quad L^q(0, T). \]

Limit $(u_0, S_0)$ solves uniquely the hysteresis (general play) RD system

\[
\partial_t u_0 - D\Delta u_0 = \left( \frac{c(S_0, N_0, F)}{c_{\min}} - 1 \right) u_0 \quad \text{a.e. in } (0, T) \times \Omega,
\]
\[
\rho_n u_0 = 0 \quad \text{a.e. in } (0, T) \times \partial \Omega,
\]
\[
u_0(0) = u_{in} \quad \text{a.e. in } \Omega,
\]
\[
\dot{S}_0(t)(S_0(t) - z) \leq 0 \quad \text{for all } \ z \in [L(f_0(t)), U(f_0(t))] \quad \text{a.e. in } [0, T],
\]
\[
S_0(0) = \min\{\max\{L(f_0(0)), S_{in}\}, U(f_0(0))\} \quad f_0(0) = F(0)/N_0(0),
\]
\[
S_0(t) \in [L(f_0(t)), U(f_0(t))] \quad \text{in } [0, T].
\]
Main theorem: singular limit to hysteresis RD system

Proof of Theorem One key problem is lack uniform bounds on $\dot{S}_\varepsilon$. This is bypassed by introducing a projection operator $p_\varepsilon$ and suitable error/monotonicity estimates.

\[
\begin{align*}
\dot{S}_\varepsilon &< 0 & \text{if } f_\varepsilon < c & \Rightarrow u(f_\varepsilon) < p_\varepsilon \\
\dot{S}_\varepsilon &= 0 & \text{if } f_\varepsilon = c & \Rightarrow S_\varepsilon = p_\varepsilon \\
\dot{S}_\varepsilon &> 0 & \text{if } f_\varepsilon > c & \Rightarrow l(f_\varepsilon) > p_\varepsilon \\
\end{align*}
\]

Figure: Sign of the gradient of $S_\varepsilon$ and projection to $p_\varepsilon$ in the $f_\varepsilon$-$S_\varepsilon$-phase diagram.
A numerical example with periodic food supply

Figure: Evolution of the population-hysteresis-diffusion system subject to the time-periodic food supply $F(t) = 0.2(1 - \cos(t))$ (blue) and the resulting population size $N$ (red) and stock $S$ (yellow).
Outline

1. Singular fast-reaction limit derivation of hysteresis RD systems

2. Nonlinear hysteresis feedback and hysteresis induced blow up

3. Weak differentiability of a class of control-to-state maps
A simplest hysteresis reaction-diffusion model

We consider

\[ \partial_t y - D \Delta y = Ry \quad \text{on} \quad \Omega \times (0, \infty), \]
\[ \partial_{\nu} y = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \]
\[ y(0) = y_0 \quad \text{on} \quad \Omega, \]

\( Ry = R(Ty, R_0) \) is a generalised scalar play operator and \( Ty \) is a projection onto spatially heterogeneous eigenfunctions \( \{ \phi_k \}_{k \geq 1} \)

\[ Ty := \underbrace{k_m \langle y, \phi_m \rangle}_{>0} + \sum_{i=m+1}^{M-1} \underbrace{k_i \langle y, \phi_i \rangle}_{\geq 0} + \underbrace{k_M \langle y, \phi_M \rangle}_{>0}, \quad 1 \leq m < M \]
Spatial homogenisation versus grow-up

\[
(T_y^{\text{cave}}(t), R^{\text{cave}}(t)) \text{ for } t \in [0,6]; (T_y^{\text{cave}}(0), R^{\text{cave}}(0)) = (1.77,1.50)
\]

\[
(T_y^{\text{vex}}(t), R^{\text{vex}}(t)) \text{ for } t \in [0,6]; (T_y^{\text{vex}}(0), R^{\text{vex}}(0)) = (1.77,1.50)
\]
Outline

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**Model**

We consider the control problem

\[ y_t - \Delta y = u + \mathcal{W}[y], \quad \text{in} \quad \Omega_T, \quad \text{(1a)} \]
\[ \mathcal{B}[y] = 0, \quad \text{on} \quad \Gamma_T, \quad \text{(1b)} \]
\[ y(\cdot, 0) = y_0, \quad \text{on} \quad \Omega. \quad \text{(1c)} \]

\( \mathcal{B} \) is a standard mixed Dirichlet Neumann boundary operators.

\( \mathcal{W} \) is a space-dependent version of a scalar operator \( \mathcal{V} \), i.e.

\[ \mathcal{W}[y](x, t) = \mathcal{V}[y(x, \cdot)](t), \quad (x, t) \in \Omega \times [0, T]. \]

Thus, \( \mathcal{W} \) represents a family of operators acting on \( y(x, \cdot) \), viewed as a function of time, at every \( x \in \Omega \).
Underlying assumptions

- **Lipschitz continuous Volterra operator**, i.e. there exists an $L > 0$:

  $$
  \mathcal{V} : \left\{ C[0, T] \rightarrow C[0, T], \right. \\
  \left. |\mathcal{V}[\nu](t) - \mathcal{V}[\tilde{\nu}](t)| \leq L \sup_{0 \leq s \leq t} |\nu(s) - \tilde{\nu}(s)|, \right.
  $$

  for every $\nu, \tilde{\nu} \in C[0, T]$ and every $t \in [0, T]$.

- **Linear growth**

  $$
  |\mathcal{V}[\nu](t)| \leq L \sup_{0 \leq s \leq t} |\nu(s)| + c_0
  $$

  for the same arguments as above and some $c_0 > 0$.

  Satisfied by many hysteresis operators, see [BS,Vis,MR].
[Visintin 1994]: System (1) has a unique solution and a well-defined control-to-state operator for any given $u \in L^2(\Omega_T)$

$$y = Su,$$

$$S : L^2(\Omega_T) \rightarrow H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V),$$

If $S$ were Fréchet differentiable, we would have for an increment $h \in L^2(\Omega_T)$

$$S(u + h) = Su + S'(u)h + o(\|h\|),$$

and $d = S'(u)h$ would be the linear first order approximation to the difference $S(u + h) - Su$. 
First order system

For given \( y = Su \) and \( h \), determine functions \( d \) and \( \omega \) as solutions of

\[
\begin{align*}
\frac{d_t}{dt} - \Delta d &= h + \omega, &\text{in } \Omega_T, \\
\omega &= \mathcal{W}'[y; d], &\text{in } \Omega_T, \\
B[d] &= 0, &\text{on } \Gamma_T, \\
d(\cdot, 0) &= 0, &\text{on } \Omega.
\end{align*}
\]

Here, \( \omega = \mathcal{W}'[y; d] \) stands for some type of derivative of \( \mathcal{W} \) at \( y \) which involves the direction \( d \).

The first order system is nonlinear if the mapping \( d \mapsto \omega \) is nonlinear.

We do not assume that the derivative depends linearly on the direction \( d \), which is not true for hysteresis operators.
Main result: Bouligand and Newton differentiability

[Brokate]: \( \mathcal{V} \) is Bouligand and Newton differentiable as operator from \( W^{1,p}(0, T) \) to \( L^r(0, T) \) for \( 1 < p < \infty \).

If \( F : O \subset X \to Y \) possesses a directional derivative \( F^{BD}(u; h) \) for all \( u \in O, h \in X \) with the property that

\[
\lim_{h \to 0} \frac{\| F[u + h] - F[u] - F^{BD}[u; h] \|}{\| h \|} = 0,
\]

then \( F \) is **Bouligand differentiable** with Bouligand derivative \( F^{BD} \).

**Theorem** The control-to-state mapping \( u \mapsto y = Su \) is Bouligand resp. Newton differentiable when considered as an operator

\[
S : L^{2+\epsilon}(0, T; L^\infty(\Omega)) \to H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; V)
\]

for sufficiently small \( \epsilon > 0 \). Moreover, the derivative is given by the solution \( d \) of the first order problem (2).
Summary:

- Hysteresis RD model as fast reaction limits
- Hysteresis diffusion driven instability
- Hysteresis RD models open for optimality conditions and semi-smooth Newton methods
- Proofs need to deal with the hysteresis non-locality in time

THANK YOU!