# Motion of a Rigid body in Compressible Fluid

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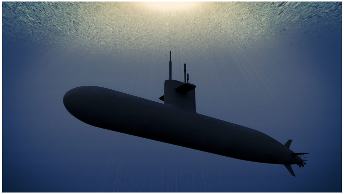
# Motivation

Studying the motion of rigid bodies inside a viscous fluid is crucial to many natural and engineering problems such as sedimentation, filtration or slurry erosion.



# Problem Description and Aim

- Motion of a rigid body in a bounded domain which is filled with a compressible isentropic fluid.
- Navier-slip boundary condition at the interface as well as at the boundary of the domain.
- $\mathcal{S}(t)$: closed, bounded and simply connected rigid body, moving in $\Omega$.
- $\mathcal{F}(t) := \Omega \setminus \mathcal{S}(t)$: filled with a viscous isentropic compressible fluid.
- Aim: Existence of a weak solution of the fluid-structure system.

# Mathematical set up:fluid equation

$$\frac{\partial \rho_F}{\partial t} + \operatorname{div}(\rho_{\mathcal{F}} u_{\mathcal{F}}) = 0 \text{ in } (0,T) \times \mathcal{F}(t),$$

$$\frac{\partial(\rho_{\mathcal{F}} u_{\mathcal{F}})}{\partial t} + \operatorname{div}(\rho_{\mathcal{F}} u_{\mathcal{F}} \otimes u_{\mathcal{F}}) - \operatorname{div} \mathbb{T}(u_{\mathcal{F}}) + \nabla p_{\mathcal{F}} = \rho_{\mathcal{F}} g_{\mathcal{F}} \text{ in } (0,T) \times \mathcal{F}(t).$$

- $\rho_{\mathcal{F}}$: mass density of the fluid, $u_{\mathcal{F}}$: velocity of the fluid.
- Isentropic flow and it is in the barotropic regime where pressure of the fluid $p_{\mathcal{F}} := a_{\mathcal{F}} \rho_{\mathcal{F}}^{\gamma}$ with $a_{\mathcal{F}} > 0$, $\gamma > \frac{3}{2}$.
- $\mathbb{T}(u_{\mathcal{F}}) = 2\mu_{\mathcal{F}} \mathbb{D}(u_{\mathcal{F}}) + \lambda_{\mathcal{F}} \operatorname{div} u_{\mathcal{F}} \mathbb{I}$, where $\mathbb{D}(u_{\mathcal{F}}) = \frac{1}{2}\left(\nabla u_{\mathcal{F}} + \nabla u_{\mathcal{F}}^{\top}\right)$, $\lambda_{\mathcal{F}}, \mu_{\mathcal{F}}$ are the viscosity coefficients satisfying

$$\mu_{\mathcal{F}} > 0, \quad 3\lambda_{\mathcal{F}} + 2\mu_{\mathcal{F}} \geqslant 0.$$

# Rigid body equation

- Rigid body: balance equations for linear and angular momentum.
- $h(t)$: centre of mass, $h'(t)$, $\omega(t)$: linear and angular velocities, $\rho_{\mathcal{S}}$: mass density.
- $u_{\mathcal{S}}(t,x) = h'(t) + \omega(t) \times (x - h(t))$.

$$mh''(t) = -\int_{\partial\mathcal{S}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}}\mathbb{I})\nu \, d\Gamma + \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} g_{\mathcal{S}} \, dx, \quad \forall\, t \in (0,T)$$

$$(J\omega)'(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t)) \times (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}}\mathbb{I})\nu \, d\Gamma, \quad \forall\, t \in (0,T)$$

$$\frac{\partial \rho_{\mathcal{S}}}{\partial t} + u_{\mathcal{S}} \cdot \nabla \rho_{\mathcal{S}} = 0 \quad \forall\, (t,x) \in (0,T) \times \mathcal{S}(t), \quad \rho_{\mathcal{S}}(0,x) = \rho_{\mathcal{S}_0}(x), \quad \forall\, x \in \mathcal{S}_0.$$

# Boundary-Initial conditions

- The slip boundary conditions on the interface of the fluid and the rigid body as well as on $\partial\Omega$.

$$
\begin{aligned}
u_{\mathcal{F}} \cdot \nu = u_{\mathcal{S}} \cdot \nu, \quad \forall\ (t,x) \in (0,T) \times \partial\mathcal{S}(t), \\
(\mathbb{T}(u_{\mathcal{F}})\nu) \times \nu = -\alpha(u_{\mathcal{F}} - u_{\mathcal{S}}) \times \nu, \quad \forall\ (t,x) \in (0,T) \times \partial\mathcal{S}(t), \\
u_{\mathcal{F}} \cdot \nu = 0, \quad \forall\ (t,x) \in (0,T) \times \partial\Omega, \\
(\mathbb{T}(u_{\mathcal{F}})\nu) \times \nu = -\alpha(u_{\mathcal{F}} \times \nu), \quad \forall\ (t,x) \in (0,T) \times \partial\Omega,
\end{aligned}
$$

$$
\begin{aligned}
\rho_{\mathcal{F}}(0,x) = \rho_{\mathcal{F}_0}(x), \quad (\rho_{\mathcal{F}} u_{\mathcal{F}})(0,x) = q_{\mathcal{F}_0}(x), \quad \forall\ x \in \mathcal{F}_0, \\
h(0) = 0, \quad h'(0) = \ell_0, \quad \omega(0) = \omega_0.
\end{aligned}
$$

# Available Literature

- Several articles on the motion of a rigid body in a viscous *incompressible* fluid.
  Weak: Serre, Desjardin-Estaben, Conca et.al, Galdi, Gunzburger, Starovoitov, Gerard-Varet, Hillairet, Sueur-Planas, Chemotov-Nečasová (up to collision), San Martin et.al, Feireisl (include collisions).
  Strong: Geissert et.al, Takahashi, Tucsnak, Wang...
- Motion of a rigid body in a viscous *compressible* fluid.
  Weak: Desjardin-Estaben (upto collision), Feireisl (include collisions).
  Strong: Boulakia-Guerrero, Roy-Takahashi, Haak et.al, Hieber-Murata.

## Main result

Let $\Omega$ and $\mathcal{S}_0 \Subset \Omega$ be two regular bounded domains of $\mathbb{R}^3$. Assume that for some $\sigma > 0$

$$\mathrm{dist}(\mathcal{S}_0, \partial\Omega) > 2\sigma.$$

Assume that the initial data satisfy

$$\rho_{\mathcal{F}_0} \in L^{\gamma}(\Omega), \quad \rho_{\mathcal{F}_0} \geqslant 0 \text{ a.e. in } \Omega, \quad \rho_{\mathcal{S}_0} \in L^{\infty}(\Omega), \quad \rho_{\mathcal{S}_0} > 0 \text{ a.e. in } \mathcal{S}_0,$$

$$q_{\mathcal{F}_0} \in L^{\frac{2\gamma}{\gamma+1}}(\Omega), \quad q_{\mathcal{F}_0} \mathbb{1}_{\{\rho_{\mathcal{F}_0}=0\}} = 0 \text{ a.e. in } \Omega, \quad \frac{|q_{\mathcal{F}_0}|^2}{\rho_{\mathcal{F}_0}} \mathbb{1}_{\{\rho_{\mathcal{F}_0}>0\}} \in L^1(\Omega),$$

$$u_{\mathcal{S}_0} = \ell_0 + \omega_0 \times x \quad \forall\, x \in \Omega \text{ with } \ell_0,\ \omega_0 \in \mathbb{R}^3.$$

Then there exists $T > 0$ (depending only on $\rho_{\mathcal{F}_0}$, $\rho_{\mathcal{S}_0}$, $q_{\mathcal{F}_0}$, $u_{\mathcal{S}_0}$, $g_{\mathcal{F}}$, $g_{\mathcal{S}}$, $\mathrm{dist}(\mathcal{S}_0, \partial\Omega)$) such that a weak solution exists on $[0, T)$. Moreover,

$$\mathcal{S}(t) \Subset \Omega, \quad \mathrm{dist}(\mathcal{S}(t), \partial\Omega) \geqslant \frac{3\sigma}{2}, \quad \forall\, t \in [0, T].$$

# Weak formulation

- $\mathcal{S}(t) \Subset \Omega$: $\chi_{\mathcal{S}}(t,x) = \mathbb{1}_{\mathcal{S}(t)}(x) \in L^\infty((0,T)\times\Omega)$.

$$u \in \left\{ \begin{aligned} & u \in L^2(0,T;L^2(\Omega)) \mid \exists\ u_{\mathcal{F}} \in L^2(0,T;H^1(\Omega)),\ u_{\mathcal{S}} \in L^2(0,T;\mathcal{R}) \\ & \text{satisfying } u(t,\cdot) = u_{\mathcal{F}}(t,\cdot) \text{ on } \mathcal{F}(t), \quad u(t,\cdot) = u_{\mathcal{S}}(t,\cdot) \text{ on } \mathcal{S}(t) \text{ with} \\ & u_{\mathcal{F}}(t,\cdot)\cdot\nu = u_{\mathcal{S}}(t,\cdot)\cdot\nu \text{ on } \partial\mathcal{S}(t),\ u_{\mathcal{F}}\cdot\nu = 0 \text{ on } \partial\Omega \text{ for a.e } t\in[0,T] \end{aligned} \right\}.$$

- $\rho \geqslant 0$, $\rho \in L^\infty(0,T;L^\gamma(\Omega))$ with $\gamma > 3/2$, $\rho|u|^2 \in L^\infty(0,T;L^1(\Omega))$, where

$$\rho = (1-\mathbb{1}_{\mathcal{S}})\rho_{\mathcal{F}} + \mathbb{1}_{\mathcal{S}}\rho_{\mathcal{S}}, \quad u = (1-\mathbb{1}_{\mathcal{S}})u_{\mathcal{F}} + \mathbb{1}_{\mathcal{S}}u_{\mathcal{S}}.$$

- Continuity, renormalized continuity equation in a weak sense.
- The transport of $\mathcal{S}$ by the rigid vector field $u_{\mathcal{S}}$.
- The continuity equation for density of the body.
- Balance of linear momentum holds in a weak sense.
- The energy inequality holds for almost every $t \in (0,T)$.

# Momentum equation

$$
\begin{aligned}
&-\int_0^T \int_{\mathcal{F}(t)} \rho_{\mathcal{F}} u_{\mathcal{F}} \cdot \frac{\partial}{\partial t}\phi_{\mathcal{F}} - \int_0^T \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} u_{\mathcal{S}} \cdot \frac{\partial}{\partial t}\phi_{\mathcal{S}} - \int_0^T \int_{\mathcal{F}(t)} (\rho_{\mathcal{F}} u_{\mathcal{F}} \otimes u_{\mathcal{F}}) : \nabla \phi_{\mathcal{F}} \\
&\quad + \int_0^T \int_{\mathcal{F}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}}\mathbb{I}) : \mathbb{D}(\phi_{\mathcal{F}}) + \alpha \int_0^T \int_{\partial\Omega} (u_{\mathcal{F}} \times \nu) \cdot (\phi_{\mathcal{F}} \times \nu) \\
&+ \alpha \int_0^T \int_{\partial\mathcal{S}(t)} [(u_{\mathcal{F}} - u_{\mathcal{S}}) \times \nu] \cdot [(\phi_{\mathcal{F}} - \phi_{\mathcal{S}}) \times \nu] = \int_0^T \int_{\mathcal{F}(t)} \rho_{\mathcal{F}} g_{\mathcal{F}} \cdot \phi_{\mathcal{F}} + \int_0^T \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} g_{\mathcal{S}} \cdot \phi_{\mathcal{S}} \\
&\qquad + \int_{\mathcal{F}(0)} (\rho_{\mathcal{F}} u_{\mathcal{F}} \cdot \phi_{\mathcal{F}})(0) + \int_{\mathcal{S}(0)} (\rho_{\mathcal{S}} u_{\mathcal{S}} \cdot \phi_{\mathcal{S}})(0).
\end{aligned}
$$

# Main Issues

- It is not possible to define a uniform velocity field due to the presence of a discontinuity through the interface of interaction.
- Not possible to write weak formulation in a global and condensed form.
- Due to the Navier-slip boundary condition, no $H^1$ bound on the velocity on the whole domain is possible.
- Usual compressible fluid issues (vanishing viscosity in the continuity equation, recovering the renormalized continuity equation, identification of the pressure).
- Passing to the limit in the transport for the rigid body is not obvious because our velocity field does not have regularity $L^\infty(0,T,L^2(\Omega))$ but we have $\sqrt{\rho}u \in L^\infty(0,T,L^2(\Omega))$.

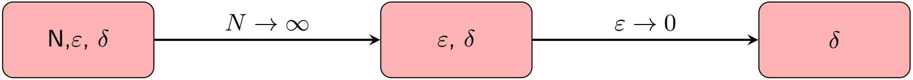
# Overcome the difficulties

- Weak formulation: need to distinguish between the fluid velocity $u_{\mathcal{F}}$ and the solid velocity $u_{\mathcal{S}}$.
- Introduce approximate problems and recover the original problem as a limit of the approximate ones.
- In each approximation levels, our solution and the test function do not show a jump across the interface so that we can use techniques of compressible fluids (without body). In the limit, the discontinuity at the interface is recovered.
- Obtain the $H^1$ regularity of the velocities of the fluid and solid parts separately. Introduce an artificial viscosity that vanishes asymptotically on the solid part.
- Isometric propagators in connection with rigid motion.

# Strategy

Approximation levels with several parameters:

- $N$: solving the momentum equation using the Faedo-Galerkin approximation.
- $\varepsilon$: a diffusion term $\varepsilon\Delta\rho$ in the continuity equation, a term $\varepsilon\nabla\rho\nabla u$ in the momentum equation.
- $\delta$: the approximation in the viscosities, a penalization of the boundary of the rigid body to get smoothness through the interface, the artificial pressure.



# Definition: Approximate solutions

$\{e_k\}_{k\geqslant 1} \subset \mathcal{D}(\overline{\Omega})$ with $e_k \cdot \nu = 0$ on $\partial\Omega$, $X_N =$ span$(e_1, \dots, e_N)$.

- $\chi_{\mathcal{S}}^N(t,x) = \mathbb{1}_{\mathcal{S}^N(t)}(x) \in L^\infty((0,T)\times\Omega) \cap C([0,T]; L^p(\Omega))$, $\forall\, 1 \leqslant p < \infty$.
- $u^N(t,\cdot) = \sum\limits_{k=1}^{N} \alpha_k(t) e_k$, $\rho^N \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$, $\rho^N > 0$ satisfies

$$\frac{\partial \rho^N}{\partial t} + \operatorname{div}(\rho^N u^N) = \varepsilon \Delta \rho^N \text{ in } (0,T)\times\Omega, \quad \frac{\partial \rho^N}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

- For all $\phi \in \mathcal{D}([0,T); X_N)$ with $\phi \cdot \nu = 0$ on $\partial\Omega$, the weak form momentum holds.
- $\chi_{\mathcal{S}}^N(t,x)$, $\rho^N \chi_{\mathcal{S}}^N(t,x)$ satisfy corresponding eqn. in the weak sense.
- Artificial pressure: $p^N(\rho) = a^N \rho^\gamma + \delta \rho^\beta$, with $a^N = a_{\mathcal{F}}(1 - \chi_{\mathcal{S}}^N)$, where $a_{\mathcal{F}}, \delta > 0$ and the exponents $\gamma$ and $\beta$ satisfy $\gamma > 3/2$, $\beta \geqslant \max\{8, \gamma\}$.
- $\mu^N = (1-\chi_{\mathcal{S}}^N)\mu_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}}^N$, $\lambda^N = (1-\chi_{\mathcal{S}}^N)\lambda_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}}^N$, $\mu^N > 0$, $2\mu^N + 3\lambda^N \geqslant 0$.
- $P_{\mathcal{S}}^N : L^2(\Omega) \to L^2(\mathcal{S}^N(t))$ is the orthogonal projection to rigid fields.

For all $\phi \in \mathcal{D}([0,T);X_N)$ with $\phi \cdot \nu = 0$ on $\partial\Omega$:

- $\int\limits_0^T \int\limits_\Omega \rho^N \left(u^N \cdot \frac{\partial}{\partial t}\phi + u^N \otimes u^N : \nabla\phi\right)$
- $\int\limits_0^T \int\limits_\Omega p^N(\rho^N)\mathbb{I} : \mathbb{D}(\phi)$
- $\alpha \int\limits_0^T \int\limits_{\partial\Omega} (u^N \times \nu) \cdot (\phi \times \nu) + \alpha \int\limits_0^T \int\limits_{\partial\mathcal{S}^N(t)} [(u^N - P_{\mathcal{S}}^N u^N) \times \nu] \cdot [(\phi - P_{\mathcal{S}}^N \phi) \times \nu]$
- $\frac{1}{\delta} \int\limits_0^T \int\limits_\Omega \chi_{\mathcal{S}}^N (u^N - P_{\mathcal{S}}^N u^N) \cdot (\phi - P_{\mathcal{S}}^N \phi).$

$$P_{\mathcal{S}}u(t,x) = \frac{1}{m}\int\limits_\Omega \rho\chi_{\mathcal{S}}u + \left(J^{-1}\int\limits_\Omega \rho\chi_{\mathcal{S}}((y-h(t)) \times u)\ dy\right) \times (x-h(t)), \quad \forall\, (t,x) \in (0,T)$$

# N, $\varepsilon$-level

- Given $u \in B_{R,T}$, solve $\rho$ (continuity equation) and $\chi_{\mathcal{S}}$ (transport of the body).
- Momentum equation: view the Galerkin approximation as a Schauder's fixed point problem. Choice of $T$.
- construction of initial data.
- convergence of $\rho^N$, $\rho^N u^N$ as in the fluid case.
- Special to body: behaviour of transport $\chi_{\mathcal{S}}^N$.

In the convergence analysis (existence of $\varepsilon$-level): most difficult terms (nonlinearity, pressure) along with the convergence of test function, energy (weak lower semicontinuity), passing in the boundary terms.

- limit of transport of the body and continuity equation.
- Momentum equation: identify the pressure ("effective viscous flux").
- Energy inequality, improved regularity of pressure (Bogovskii).

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# $\delta$-level

- $\sqrt{\chi_{\mathcal{S}}^{\delta}}\left(u^{\delta}-P_{\mathcal{S}}^{\delta}u^{\delta}\right)\to 0$ strongly in $L^2((0,T)\times\Omega)$.
- convergence: $\chi_{\mathcal{S}}\left(u-u_{\mathcal{S}}\right)=0$.
- velocity field in the fluid part: $u_{\mathcal{F}}^{\delta}(t,\cdot)=\mathcal{E}_u^{\delta}(t)\left[u^{\delta}(t,\cdot)|_{\mathcal{F}^{\delta}}\right]$ such that

  $\{u_{\mathcal{F}}^{\delta}\}$ is bounded in $L^2(0,T;H^1(\Omega))$, $\quad u_{\mathcal{F}}^{\delta}=u^{\delta}$ on $\mathcal{F}^{\delta}$, i.e. $(1-\chi_{\mathcal{S}}^{\delta})(u^{\delta}-u_{\mathcal{F}}^{\delta})=0$.

- Identify $\rho_{\mathcal{S}}$, $\rho_{\mathcal{F}}$.
- Momentum equation: construction of test function space-capture the discontinuity in the limit. Construct $\phi_{\mathcal{S}}^{\delta}$ of $\phi$ without jumps at the interface such that

$$\phi_{\mathcal{S}}^{\delta}(t,x)=\phi_{\mathcal{F}}(t,x)\quad\forall\ t\in(0,T),\ x\in\partial\mathcal{S}^{\delta}(t),$$

  $\phi_{\mathcal{S}}^{\delta}(t,\cdot)\approx\phi_{\mathcal{S}}(t,\cdot)$ in $\mathcal{S}^{\delta}(t)$ away from a $\delta^{\vartheta}$ neighborhood of $\partial\mathcal{S}^{\delta}(t)$ with $\vartheta>0$.

- Use the test function in passing the pressure term.
- Energy inequality and the body is away from boundary.

Thank You