Discrete Fuglede conjecture on cyclic groups

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Fuglede's conjecture in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set with $\lambda(\Omega) > 0$.
We say that

- Ω is a *tile*, if $\exists T \subset \mathbb{R}^n$ s. t. λ-a.a. x can be uniquely written as the sum of an element of Ω and an element of T.
  T is a *tiling complement* of Ω.
- Ω is *spectral*, if there is a base of $L^2(\Omega)$ consisting only of exponential functions $\{f(x) = e^{2\pi i<x,\lambda>} | \lambda \in \Lambda\}$.
  Λ is called a *spectrum* for Ω.

Conjecture (Fuglede's conjecture (1974))

The spectral sets are the tiles in $\mathbb{R}^n$.

Historical background

Theorem (Fuglede '74)

If Ω is a tile with a tiling complement, which is a lattice, then Ω is spectral.

After some positive results T. Tao [10] disproved the conjecture.

Theorem (Tao '04)

Fuglede's conjecture fails in $\mathbb{R}^n$ if $n \geq 5$. There exists a spectral set that is not a tile.

This was improved in two ways.

- There were found some non-tiling spectral sets in $\mathbb{R}^n$ for $n \geq 3$ by M. Kolountzakis and M. Matolcsi [6].
- There were shown non-spectral tiles in $\mathbb{R}^n$ for $n \geq 3$ by B. Farkas, M. Matolcsi and P. Móra [2].

Both directions of the conjecture are still open in $\mathbb{R}$ and $\mathbb{R}^2$.

Spectral sets and tiles in finite Abelian groups

Let G be a finite Abelian group and $\widehat{G}$ the set of irreducible representations of G. Well-known that $G \simeq \widehat{G}$.
The elements of $\widehat{G}$ can be indexed by the elements of G.

- $S \subset G$ is *spectral* if there exists a $\Lambda \subset G$ such that $(\chi_l)_{l \in \Lambda}$ is an orthogonal base of complex valued functions defined on S.
- If Λ is a spectrum for S, then S is a spectrum for Λ. We say that (S, Λ) is a *spectral pair* and $|S| = |\Lambda|$.
- $S \subset G$ is *a tile* of G if there is a $T \subset G$ such that $S + T = G$ and $|S| \cdot |T| = |G|$. We denote this by $S \bigoplus T$.

Fuglede's conjecture on finite Abelian groups

Conjecture (Discrete Fuglede's conjecture)

Let G be an Abelian group. Then the spectral sets and the tiles coincide.

All counterexamples $\mathbb{R}^n$ are based on counterexamples for the discrete version of Fuglede's conjecture.

- Tao [10] proved that in $(\mathbb{Z}_2)^{11}$ and in $(\mathbb{Z}_3)^5$ there is a non-tiling spectral set.
- Kolounztakis and Matolcsi [6] showed a non-tiling spectral set in $(\mathbb{Z}_8)^3$.
- Farkas, Matolcsi, Móra [2] showed a non-spectral tile in $(\mathbb{Z}_{24})^3$.

Question

For which groups does Fuglede's conjecture hold?

Connections of cyclic groups and one dimension

$T - S(G)$: Tile $\Rightarrow$ spectral direction holds on G
$S - T(G)$: Spectral $\Rightarrow$tile direction holds on G.
Dutkay and Lai [1] proved the following:

$$T - S(\mathbb{R}) \Leftrightarrow T - S(\mathbb{Z}) \Leftrightarrow T - S(\mathbb{Z}_{\mathbb{N}}),$$
$$S - T(\mathbb{R}) \Rightarrow S - T(\mathbb{Z}) \Rightarrow S - T(\mathbb{Z}_{\mathbb{N}}),$$

where $T - S(\mathbb{Z}_{\mathbb{N}})$ means that $T - S(\mathbb{Z}_n)$ holds for every $n \in \mathbb{N}$.

Positive results for cyclic groups

The conjecture holds for

- $\mathbb{Z}_{p^n}$, where p is a prime and $n \in \mathbb{N}$,
- $\mathbb{Z}_{p^n q^k}$, where p, q are primes, $n, k \in \mathbb{N}$ and $\min(n, k) \leq 6$ by [8],
- $\mathbb{Z}_{pqr}, \mathbb{Z}_{p^2 qr}$, where p, q, r are primes.

Theorem (K.-Malikiosis-Somlai-Vizer, [5])

Fuglede's conjecture holds for $\mathbb{Z}_{pqrs}$, where p, q, r, s are primes.

Conjecture

Fuglede's conjecture holds for every cyclic group.

Tiles of cyclic groups

Lemma (Tijdeman's dilation lemma)

Let $A \bigoplus B = \mathbb{Z}_N$.

- *If* p *is a prime such that* $p \nmid |A|$, *then* $pA \bigoplus B = \mathbb{Z}_N$.
- *If* $(k, |A|) = 1$, *then* $kA \bigoplus B = \mathbb{Z}_N$.

Lemma (Sand)

$A \bigoplus B = \mathbb{Z}_N$ *if and only if* $N = |A||B|$ *and the subsets* $A - A$ *and* $B - B$ *contain no non-zero elements of the same order.*

Cowen-Meyerowitz: Property (T1) and (T2)

To a set $S \subseteq \mathbb{Z}_N$ one can associate a polynomial $m_S(x)$, called the **mask polynomial of** S, defined as $m_S(x) = \sum_{s \in S} x^s$. Let H_S be the set of prime powers r^a dividing N such that $\Phi_{r^a}(x) \mid m_S(x)$.

(T1) $m_S(1) = \prod_{d \in H_S} \Phi_d(1)$.

(T2) For pairwise relative prime elements s_i of H_S, $\Phi_{\prod s_i} \mid m_S(x)$.

Theorem

Let $S \subseteq \mathbb{Z}_N$. Then the following statements hold.

1. *If S satisfies (T1) and (T2), then S tiles $\mathbb{Z}_N$.*
2. *If S tiles the $\mathbb{Z}_N$, then S satisfies (T1).*
3. *If S satisfies (T1) and (T2), then S is a spectral set.*

Spectral sets in $\mathbb{Z}_N$

Let $S \subseteq \mathbb{Z}_N$ be a spectral set with spectrum Λ. (i.e. (S, Λ) is a spectral pair.)
Reformulation of spectrality: $|S| = |\Lambda|$ and

1. $\Lambda - \Lambda \subseteq \{0\} \cup \{x \in \mathbb{Z}_N \colon \hat{1}_S(x) = 0\}$, where 1_S is the characteristic function of S, and $\hat{f}(x) = \sum_{y \in \mathbb{Z}_N} f(y) \xi_N^{-xy}$ is the (discrete) Fourier transform of $f : G \to \mathbb{C}$.

1'. $\Lambda - \Lambda \subseteq \{0\} \cup \{d \mid N \in \mathbb{N} \colon m_S(\xi_d) = 0\}$.

Note that : if $g \in \mathbb{Z}_N^{\star}$, then $m_S(\xi_d) = 0 \Rightarrow m_S(\xi_d^g) = 0$.
Thus

$$m_S(\xi_d) = 0 \iff \Phi_d \mid m_S.$$

Equidistributivity property

Let $m_S(\xi_p) = 0$ ($\Leftrightarrow \Phi_p \mid m_S$).
The minimal polynomial of $\xi_p = e^{\frac{2\pi i}{p}}$ over $\mathbb{Q}$ is $\sum_{j=0}^{p-1} x^j$. It implies that the sets $S_k := \{u \in S \colon u \equiv k \pmod{p}\}$ satisfies

$$|S_k| = \frac{|S|}{p}$$

for each $k = 0, \ldots, p-1$.

Corollary

If $d \in \Lambda - \Lambda$ with $o(d) = p$, then $p \mid |S| = |\Lambda|$.

Cube rule I.

Let $m_S(\xi_n) = 0$ for some $n = p_1 \cdots p_k \mid N$ ($\Leftrightarrow \Phi_n \mid m_S$).
Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \bigoplus \cdots \bigoplus \mathbb{Z}_{p_k} \leq \mathbb{Z}_N$ can be taken as a subset of the k-dimensional integer grid and $S_{\mathbb{Z}_n}$ denote the projection of S to $\mathbb{Z}_n$. A subset C of $\mathbb{Z}_n$ an *k-dimensional cube*, if $C = \bigoplus_{i=1}^{k} A_i$, where $A_i \subset \mathbb{Z}_{p_i}$ with $|A_i| = 2$.

Lemma (Cube rule, [4])

Let n and N as above and $m_S(\xi_n) = 0$. Then for every k-dimensional cube C and $c_0 \in C$ the following hold

$$\sum_{c \in C} (-1)^{d_H(c_0,c)} S_{\mathbb{Z}_n}(c) = 0.$$

Equivalently, if $m_S(\xi_n) = 0$ then $S_{\mathbb{Z}_n}$ is the weighted sum of $\mathbb{Z}_{p_1}$-,..., $\mathbb{Z}_{p_k}$-cosets with rational coefficients.

Cube rule II.

- Cube rule implies equidistributivity for $n = p$.
- For $n = p_1 \cdot p_2$ we can more specific:

Lemma (Lam and Leung [7])

If $n = p_1 \cdot p_2$ then $m_S(\xi_n) = 0$ implies that the multiset $S_{\mathbb{Z}_n}$ is the weighted sum of $\mathbb{Z}_{p_1}$- and $\mathbb{Z}_{p_2}$-cosets with **nonnegative integer coefficients.**

Tile-spectral direction on $\mathbb{Z}_N$ for squarefree N

The following result was realized by I. Łaba and A. Meyerowitz, and rediscovered by R. Shi [9].

Lemma

Let N be squarefree and A, B sets such that $A \bigoplus B = \mathbb{Z}_N$, where $|A| = k$. Then $A \bigoplus kB = \mathbb{Z}_N$ and kB is a subgroup.

The proof is based on Tijdeman's result: if $(p, |B|) = 1$, then $A \bigoplus pB = \mathbb{Z}_N$ and pB is the set where we forgot the p-th coordinate of B.

It simply follows from this lemma that conditions (T1) and (T2) holds for B, which implies that B is spectral. Similarly, A is spectral.

Spectral-tile direction on $\mathbb{Z}_N$ for squarefree N

Lemma

Let (S, Λ) be a spectral pair in $\mathbb{Z}_N$. If S or Λ is the union of $\mathbb{Z}_p$-cosets, then S is a tile.

In some cases we show that if $S \subset \mathbb{Z}_N$ is spectral, then it is a tile:

- If $|S| = 1$ or $|S| = N$, then it is trivial.
- If $N = p$ and $|S| > 1$, then $\Phi_p \mid m_S$ hence $p \mid |S|$. Thus $p = |S|$.
- If $N = p_1 p_2$ and $|S| > 1$, then we have the following cases:
  1. $\Phi_{p_1}\Phi_{p_2} \mid m_S \Longrightarrow |S| = p_1 p_2$
  2. $\Phi_{p_1 p_2} \mid m_S$, then by the cube rule S is the union of $\mathbb{Z}_{p_1}$-cosets (or $\mathbb{Z}_{p_2}$-cosets.)
  3. $\Phi_{p_1} \nmid m_S$ and $\Phi_{p_1 p_2} \nmid m_S$, then $\Phi_{p_2} \mid m_S$ (i.e. $p_2 \mid |S|$ and S is equidistributed) and every nonzero element of $\Lambda - \Lambda$ is of order p_2, hence $|S| = p_2$ and S is a tile.

Spectral-tile direction on $\mathbb{Z}_N$ for squarefree N

If $N = p_1 p_2 p_3$ and $|S| > 1$, then we distinguish two cases.

- If $\Phi_{p_1p_2p_3} \nmid m_S$ then either $|S| = p_i$ and S is equidistributed or we can apply the 2-dimensional cube rule, which reduce the problem to the previous case.
- If $\Phi_{p_1p_2p_3} \mid m_S$, then we apply 3-dimensional cube rule.
  1. If $d_H(x, y) = 1$ for all $x, y \in S$, then S is a $\mathbb{Z}_{p_i}$-coset.
  2. If $d_H(x, y) = 3$ for all $x, y \in S$, then by 3d cube rule we get a contradiction.
  3. If $d_H(x, y) = 2$ for some $x, y \in S$, then by 3d cube rule we get that S is the union of Z_{p_i}-cosets.

The result for $N = p_1 p_2 p_3 p_4$ is similar case-by-case argument, but much more complicated.

Some particular open problem

Question

Does Fuglede's conjecture hold for

1. $\mathbb{Z}_N$, *where* N *is squarefree (e.g.* $N = p_1 \cdots p_5$*)?*
2. $\mathbb{Z}_{p^n q^k}$, *where* p, q *are different primes and* $n, k \in \mathbb{N}$*?*

Thank you for your kind attention.

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