Maximal Noncompactness of Sobolev Embeddings

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Maximal Noncompactness of Sobolev Embeddings

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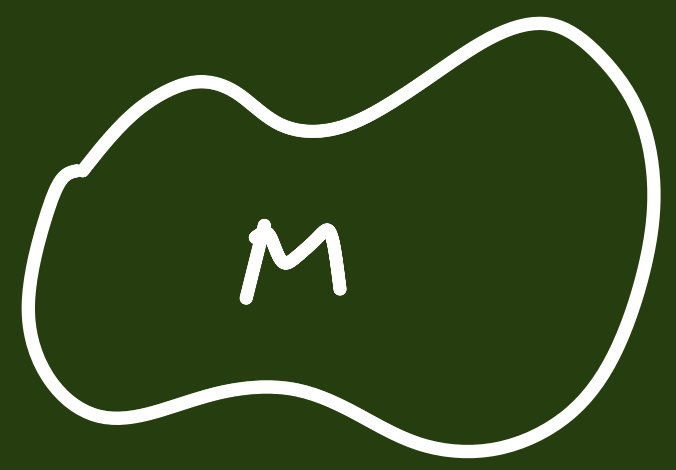
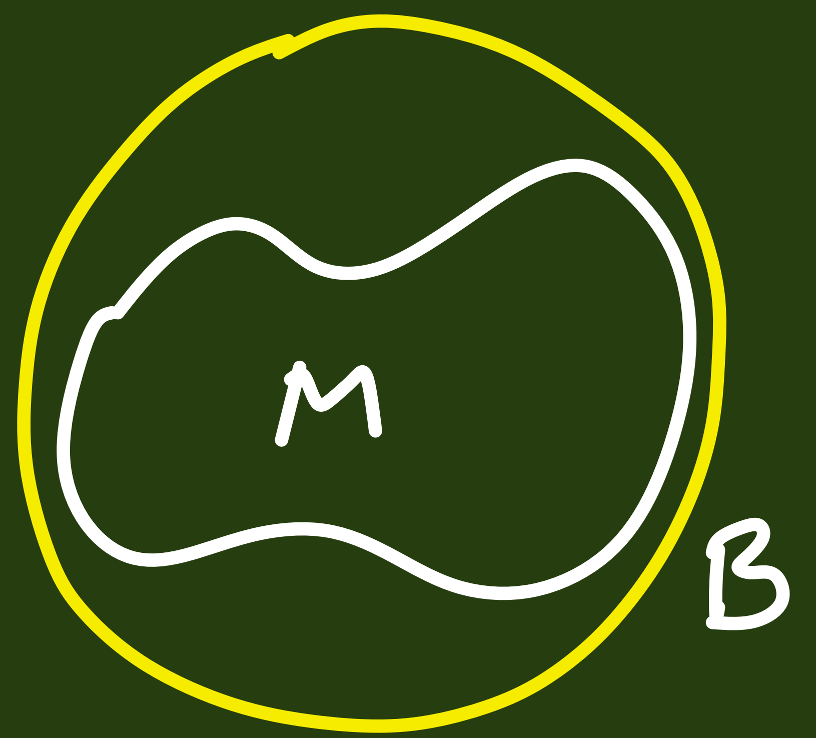
Noncompactness

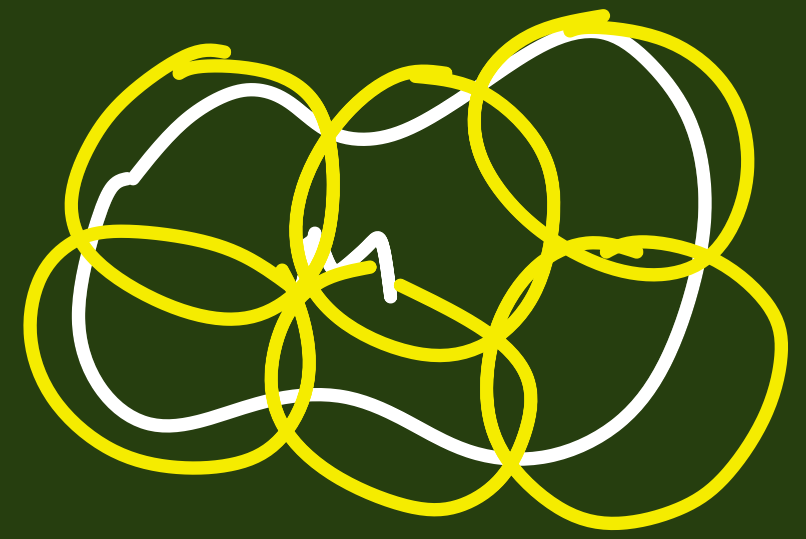
let M be a set in a metric space



- M is bounded



- M is compact



- (ball) measure of noncompactness

$$\alpha(M) = \inf\{\rho : \exists \text{ finite set of balls with radius } \rho \text{ covering } M\}$$

- For a linear mapping $T: X \to Y$, the same notions are defined if we set $M = T[B_X]$

- Trivial estimates

$$0 \leq \alpha(T) \leq \|T\|$$

- Extreme cases

$\alpha(T) = 0$ ... T is compact

$\alpha(T) = \|T\|$ ... T is maximally noncompact

Sobolev Embeddings

$\Omega \subset \mathbb{R}^n$ open bounded

$$I : V_0^k X(\Omega) \to Y(\Omega)$$

- $u : \Omega \to \mathbb{R}$, continuation by 0 is k-times weak. diff. in $\mathbb{R}^n$

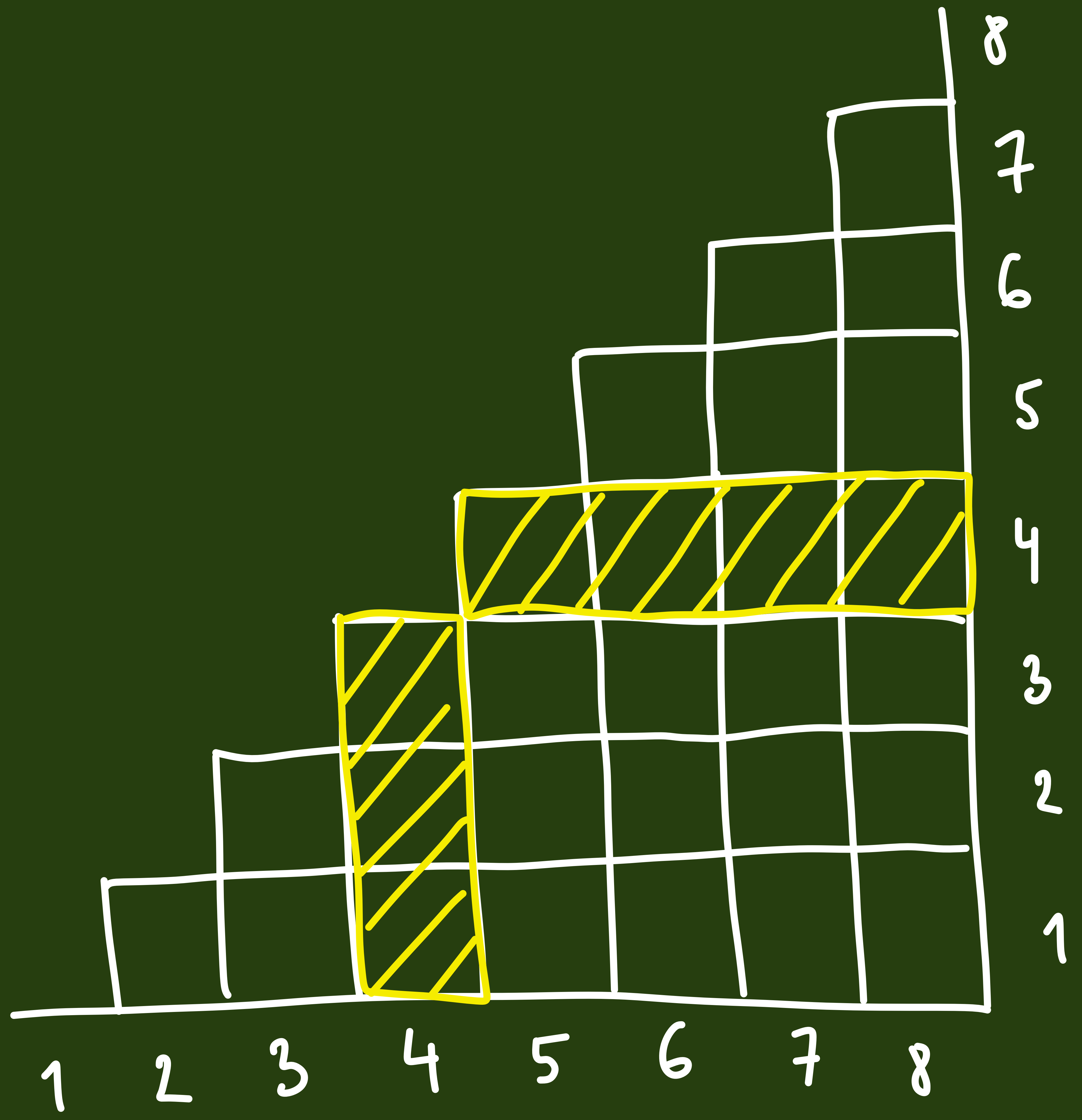
$$\|u\|_{V_0^k X(\Omega)} = \sum_{|\beta| = k} \|D^\beta u\|_{X(\Omega)}$$

- X, Y Lebesgue & Lorentz spaces

... refinement of Lebesgue scale ...

$$\begin{tikzcd} & L^1 & & \\ L^{p,1} \arrow[r] & L^p = L^{p,p} \arrow[u] \arrow[r] & L^{p,\infty} \\ & L^\infty \arrow[u] & & \end{tikzcd}$$

Intermezzo (coloring problem)



- side length $= 2^k$
- color the triangle s.t. j-th row & j-th column do not share a color
- Minimum colors needed?

Back to Embeddings

(all spaces over Ω)

$$I : V_0^k L^p \to L^{p^*} \qquad p \in [1, \tfrac{n}{k}), \quad p^* = \frac{np}{n-kp}$$

- noncompact
- maximally noncompact (Hencl, Math. Nachr. 2003)

$$I : V_0^k L^p \to L^{p^*, p}$$

- sharp ($p^* > p$), noncompact
- maximally noncompact (Bouchala, Z. Anal. Anwend. 2020)

$$I : V_0^k L^{p,q} \to L^{p^*,s} \qquad 1 \le q \le s < \infty$$

- the same (still Bouchala)

$$I : V_0^k L^{p} \to L^{p^*,\infty}$$

- weaker, still noncompact
- is it maximally noncompact ?

Disjoint superadditivity

$(\exists \gamma > 0)(\exists C > 0)(\forall m \in \mathbb{N})(\forall f_1, \dots, f_m$ disjointly supported$)$

$$\sum_{k=1}^{m} \|f_k\|_X^{\gamma} \leq C \left\| \sum_{k=1}^{m} f_k \right\|_X^{\gamma}$$

- L^p, $L^{p,q}$ are disjointly superaditive for $p, q < \infty$

Proposition $L^{p,\infty}$ is not.

Shrinking property

$I : X(\Omega) \to Y(\Omega)$ is shrinking if

$$\|I\| = \sup\{ \|u\|_{Y(\Omega)} : \|u\|_X \leq 1 \quad \operatorname{supp} u \subset G \}$$

for every open $G \subset \Omega$.

Theorem

$I : X \to L^{r,\infty}$ is shrinking $\Rightarrow$ I is maximally noncompact

Corollary

$I : V_0^K L^{p,q} \to L^{p^*,\infty}$ is shrinking

Therefore maximally noncompact

Limiting case $\quad Y = L^\infty$

$$I : V_0^k L^{\frac{n}{k},1} \to L^\infty$$

- sharp, noncompact

- what is $\alpha(I)$?

Span

$$I : X(\Omega) \to L^\infty(\Omega)$$

$$\sigma(I) = \sup \left\{ \operatorname*{ess\,sup}_{\Omega} u - \operatorname*{ess\,inf}_{\Omega} u \ : \ \|u\|_{X(\Omega)} \le 1 \right\}$$

Clearly $\cdot$ $\sigma(I) \le 2\|I\|$

Often $\cdot$ $\sigma(I) \ge \|I\|$ $\quad (\Leftarrow$ shrinking$)$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}} \\ \sigma(I) \ge \|I\| \quad \Rightarrow \quad \alpha(I) \le \dfrac{\sigma(I)}{2}\end{array}}$$

$$I : V_0^k L^{\frac{n}{k},1} \longrightarrow L^\infty$$

Proposition $\quad I$ is shrinking

Proposition $\quad \sigma(I) = 2^{1-\frac{k}{n}} \|I\|$

Corollary $\quad \alpha(I) \leq 2^{-\frac{k}{n}} \|I\|$

i.e. I is not maximally noncompact

In fact: $\quad \alpha(I) = 2^{-\frac{k}{n}} \|I\|$

Theorem Let $\rho > 0$. Suppose that for any $\ell \in \mathbb{N}$, there are $u_k \in X$ s.t.

- $\operatorname{suppt} u_j \cap \operatorname{suppt} u_k = \emptyset \qquad j \neq k$
- $\| u_j - u_k \|_X \leq 1 \qquad j, k = 1, \dots, \ell$
- $2\rho > \operatorname{esssup}_{\Omega} u_k > \rho \qquad k = 1, \dots, \ell$

$$\Longrightarrow \quad \alpha(I : X \to L^\infty) \geq \rho$$

That's all Folks!