

Weighted Energy-Dissipation principle for nonlinear stochastic evolution equations

Luca Scarpa

Department of Mathematics
Politecnico di Milano
luca.scarpa@polimi.it

8th European Congress of Mathematics
Minisymposium on Stochastic Evolution Equations

20-26 June 2021, Portorož, Slovenia



POLITECNICO
MILANO 1863



**universität
wien**

FWF

Der Wissenschaftsfonds.

- 1 WED principle for deterministic evolution equations
- 2 WED principle for stochastic evolution equations
- 3 Future developments

General variational setting

We consider evolution equations in the form

$$\partial_t u(t) + \partial\phi(u(t)) \ni 0 \quad \text{in } H, \quad t \in [0, T], \quad u(0) = u_0,$$

where:

- H is a real separable Hilbert space, $T > 0$
- $\phi : H \rightarrow [0, +\infty]$: proper, convex, lower semicontinuous on H
- $\partial\phi : H \rightarrow 2^H$ subdifferential of ϕ
- $u_0 \in H$: initial datum

Ideas of the Weighted-Energy-Dissipation (WED) principle:

- introduction of global-in-time convex functionals I_ε on trajectories, with $\varepsilon > 0$ positive parameter
- I_ε admits a unique minimizer u_ε
- u_ε is a solution to an elliptic-in-time regularization of the equation
- $u_\varepsilon \rightarrow u$ in suitable topologies

General variational setting

We consider evolution equations in the form

$$\partial_t u(t) + \partial\phi(u(t)) \ni 0 \quad \text{in } H, \quad t \in [0, T], \quad u(0) = u_0,$$

where:

- H is a real separable Hilbert space, $T > 0$
- $\phi : H \rightarrow [0, +\infty]$: proper, convex, lower semicontinuous on H
- $\partial\phi : H \rightarrow 2^H$ subdifferential of ϕ
- $u_0 \in H$: initial datum

Ideas of the Weighted-Energy-Dissipation (WED) principle:

- introduction of global-in-time convex functionals I_ε on trajectories, with $\varepsilon > 0$ positive parameter
- I_ε admits a unique minimizer u_ε
- u_ε is a solution to an elliptic-in-time regularization of the equation
- $u_\varepsilon \rightarrow u$ in suitable topologies

The WED principle for deterministic equations

Form of WED functionals:

$$I_\varepsilon(v) := \int_0^T e^{-t/\varepsilon} \left[\frac{\varepsilon}{2} \|\partial_t v(t)\|_H^2 + \phi(v(t)) \right] dt$$

if $v \in H^1(0, T; H)$, $\phi(v) \in L^1(0, T)$, $v(0) = u_0$,

$$I_\varepsilon(v) := +\infty \quad \text{otherwise.}$$

Regularized equation satisfied by the minimizer u_ε of I_ε :

$$-\varepsilon \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + \partial \phi(u_\varepsilon) \ni 0, \quad u_\varepsilon(0) = u_0, \quad \varepsilon \partial_t u_\varepsilon(T) = 0.$$

Main advantages:

- approximability of nonlinear equations with convex minimization problems
- convex minimization is usually easier to handle
- relation to analytical theory and numerical approximation

The WED principle for deterministic equations

Form of WED functionals:

$$I_\varepsilon(v) := \int_0^T e^{-t/\varepsilon} \left[\frac{\varepsilon}{2} \|\partial_t v(t)\|_H^2 + \phi(v(t)) \right] dt$$

if $v \in H^1(0, T; H)$, $\phi(v) \in L^1(0, T)$, $v(0) = u_0$,

$$I_\varepsilon(v) := +\infty \quad \text{otherwise.}$$

Regularized equation satisfied by the minimizer u_ε of I_ε :

$$-\varepsilon \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + \partial \phi(u_\varepsilon) \ni 0, \quad u_\varepsilon(0) = u_0, \quad \varepsilon \partial_t u_\varepsilon(T) = 0.$$

Main advantages:

- approximability of nonlinear equations with convex minimization problems
- convex minimization is usually easier to handle
- relation to analytical theory and numerical approximation

The WED principle for deterministic equations

Form of WED functionals:

$$I_\varepsilon(v) := \int_0^T e^{-t/\varepsilon} \left[\frac{\varepsilon}{2} \|\partial_t v(t)\|_H^2 + \phi(v(t)) \right] dt$$

if $v \in H^1(0, T; H)$, $\phi(v) \in L^1(0, T)$, $v(0) = u_0$,

$$I_\varepsilon(v) := +\infty \quad \text{otherwise.}$$

Regularized equation satisfied by the minimizer u_ε of I_ε :

$$-\varepsilon \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + \partial \phi(u_\varepsilon) \ni 0, \quad u_\varepsilon(0) = u_0, \quad \varepsilon \partial_t u_\varepsilon(T) = 0.$$

Main advantages:

- approximability of nonlinear equations with convex minimization problems
- convex minimization is usually easier to handle
- relation to analytical theory and numerical approximation

What happens in the stochastic case?

General form of nonlinear stochastic equation:

$$du + \partial\phi(u) dt \ni B dW, \quad u(0) = u_0,$$

where:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ filtered probability space
- V separable reflexive Banach space, H Hilbert space: $V \hookrightarrow H \hookrightarrow V^*$
- W : cylindrical Wiener process on a Hilbert space U
- $B \in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))$: stochastically integrable Hilbert-Schmidt coefficient
- $\partial\phi : V \rightarrow 2^{V^*}$ and $p \geq 2$ such that

$$\langle w, z \rangle_{V^*, V} \geq c \|z\|_V^p, \quad \|w\|_{V^*} \leq C(1 + \|z\|_V^{p-1}) \quad \forall (z, w) \in \partial\phi$$

- $u_0 \in H$: initial datum

In this setting, the equation admits a unique solution (Pardoux, Krylov-Rozovskiĭ)

$$u \in L^2(\Omega; C^0([0, T]; H)) \cap L^p(\Omega; L^p(0, T; V)).$$

QUESTION: WED principle for SPDEs?

What happens in the stochastic case?

General form of nonlinear stochastic equation:

$$du + \partial\phi(u) dt \ni B dW, \quad u(0) = u_0,$$

where:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ filtered probability space
- V separable reflexive Banach space, H Hilbert space: $V \hookrightarrow H \hookrightarrow V^*$
- W : cylindrical Wiener process on a Hilbert space U
- $B \in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))$: stochastically integrable Hilbert-Schmidt coefficient
- $\partial\phi : V \rightarrow 2^{V^*}$ and $p \geq 2$ such that

$$\langle w, z \rangle_{V^*, V} \geq c \|z\|_V^p, \quad \|w\|_{V^*} \leq C(1 + \|z\|_V^{p-1}) \quad \forall (z, w) \in \partial\phi$$

- $u_0 \in H$: initial datum

In this setting, the equation admits a unique solution (Pardoux, Krylov-Rozovskiĭ)

$$u \in L^2(\Omega; C^0([0, T]; H)) \cap L^p(\Omega; L^p(0, T; V)).$$

QUESTION: WED principle for SPDEs?

What happens in the stochastic case?

General form of nonlinear stochastic equation:

$$du + \partial\phi(u) dt \ni B dW, \quad u(0) = u_0,$$

where:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ filtered probability space
- V separable reflexive Banach space, H Hilbert space: $V \hookrightarrow H \hookrightarrow V^*$
- W : cylindrical Wiener process on a Hilbert space U
- $B \in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))$: stochastically integrable Hilbert-Schmidt coefficient
- $\partial\phi : V \rightarrow 2^{V^*}$ and $p \geq 2$ such that

$$\langle w, z \rangle_{V^*, V} \geq c \|z\|_V^p, \quad \|w\|_{V^*} \leq C(1 + \|z\|_V^{p-1}) \quad \forall (z, w) \in \partial\phi$$

- $u_0 \in H$: initial datum

In this setting, the equation admits a unique solution (Pardoux, Krylov-Rozovskiĭ)

$$u \in L^2(\Omega; C^0([0, T]; H)) \cap L^p(\Omega; L^p(0, T; V)).$$

QUESTION: WED principle for SPDEs?

What happens in the stochastic case?

General form of nonlinear stochastic equation:

$$du + \partial\phi(u) dt \ni B dW, \quad u(0) = u_0,$$

where:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ filtered probability space
- V separable reflexive Banach space, H Hilbert space: $V \hookrightarrow H \hookrightarrow V^*$
- W : cylindrical Wiener process on a Hilbert space U
- $B \in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))$: stochastically integrable Hilbert-Schmidt coefficient
- $\partial\phi : V \rightarrow 2^{V^*}$ and $p \geq 2$ such that

$$\langle w, z \rangle_{V^*, V} \geq c \|z\|_V^p, \quad \|w\|_{V^*} \leq C(1 + \|z\|_V^{p-1}) \quad \forall (z, w) \in \partial\phi$$

- $u_0 \in H$: initial datum

In this setting, the equation admits a unique solution (Pardoux, Krylov-Rozovskiĭ)

$$u \in L^2(\Omega; C^0([0, T]; H)) \cap L^p(\Omega; L^p(0, T; V)).$$

QUESTION: WED principle for SPDEs?

What happens in the stochastic case?

General form of nonlinear stochastic equation:

$$du + \partial\phi(u) dt \ni B dW, \quad u(0) = u_0,$$

where:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ filtered probability space
- V separable reflexive Banach space, H Hilbert space: $V \hookrightarrow H \hookrightarrow V^*$
- W : cylindrical Wiener process on a Hilbert space U
- $B \in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H)))$: stochastically integrable Hilbert-Schmidt coefficient
- $\partial\phi : V \rightarrow 2^{V^*}$ and $p \geq 2$ such that

$$\langle w, z \rangle_{V^*, V} \geq c \|z\|_V^p, \quad \|w\|_{V^*} \leq C(1 + \|z\|_V^{p-1}) \quad \forall (z, w) \in \partial\phi$$

- $u_0 \in H$: initial datum

In this setting, the equation admits a unique solution (Pardoux, Krylov-Rozovskiĭ)

$$u \in L^2(\Omega; C^0([0, T]; H)) \cap L^p(\Omega; L^p(0, T; V)).$$

QUESTION: WED principle for SPDEs?

The WED functionals in the stochastic case

Reformulation of the equation: we look for an Itô-type process

$$u(t) = u^d(t) + \int_0^t u^s(s) dW(s), \quad t \in [0, T],$$

such that

$$\begin{cases} \partial_t u^d(t) + \partial \phi(u(t)) \ni 0 & t \in [0, T], \\ u^s = B, \\ u(0) = u_0. \end{cases}$$

Idea for the WED functionals in the stochastic case:

$$I_\varepsilon(v) := \mathbb{E} \int_0^T e^{-t/\varepsilon} \left[\frac{\varepsilon}{2} \|\partial_t v^d(t)\|_H^2 + \phi(v(t)) + \frac{1}{2} \|(v^s - B)(t)\|_{\mathcal{L}^2}^2 \right] dt$$

if $v^d \in L^2(\Omega; H^1(0, T; H))$, $\phi(v) \in L^1(\Omega \times (0, T))$, $v(0) = u_0$,

$$I_\varepsilon(v) := +\infty \quad \text{otherwise.}$$

The WED functionals in the stochastic case

Reformulation of the equation: we look for an Itô-type process

$$u(t) = u^d(t) + \int_0^t u^s(s) dW(s), \quad t \in [0, T],$$

such that

$$\begin{cases} \partial_t u^d(t) + \partial \phi(u(t)) \ni 0 & t \in [0, T], \\ u^s = B, \\ u(0) = u_0. \end{cases}$$

Idea for the WED functionals in the stochastic case:

$$I_\varepsilon(v) := \mathbb{E} \int_0^T e^{-t/\varepsilon} \left[\frac{\varepsilon}{2} \|\partial_t v^d(t)\|_H^2 + \phi(v(t)) + \frac{1}{2} \|(v^s - B)(t)\|_{\mathcal{L}^2}^2 \right] dt$$

if $v^d \in L^2(\Omega; H^1(0, T; H))$, $\phi(v) \in L^1(\Omega \times (0, T))$, $v(0) = u_0$,

$$I_\varepsilon(v) := +\infty \quad \text{otherwise.}$$

The Euler–Lagrange equation in the stochastic case

Regularized equation at $\varepsilon > 0$:

$$\begin{cases} -\varepsilon \partial_t (\partial_t u_\varepsilon^d)^d + \partial_t u_\varepsilon^d + \partial \phi(u_\varepsilon) \ni 0, \\ u_\varepsilon^s = B + \varepsilon (\partial_t u_\varepsilon^d)^s, \\ \varepsilon \partial_t u_\varepsilon^d(T) = 0, \\ u_\varepsilon(0) = u_0. \end{cases}$$

Setting $v_\varepsilon := \partial_t u_\varepsilon^d$, this equivalent to a forward-backward stochastic system:

$$\begin{cases} du_\varepsilon = v_\varepsilon dt + (B + \varepsilon G_\varepsilon) dW, \\ u_\varepsilon(0) = u_0, \end{cases} \quad \begin{cases} -\varepsilon dv_\varepsilon + v_\varepsilon dt + \partial \phi(u_\varepsilon) dt \ni -\varepsilon G_\varepsilon dW, \\ \varepsilon v_\varepsilon(T) = 0. \end{cases}$$

Difficulties:

- well-posedness of forward-backward system (ε fixed): backward stochastic equation
- equivalence of the Euler-Lagrange system (ε fixed): martingale representation theorems
- passage to the limit ($\varepsilon \rightarrow 0^+$): requires a forward-backward version of Itô's formula

The Euler–Lagrange equation in the stochastic case

Regularized equation at $\varepsilon > 0$:

$$\begin{cases} -\varepsilon \partial_t (\partial_t u_\varepsilon^d)^d + \partial_t u_\varepsilon^d + \partial \phi(u_\varepsilon) \ni 0, \\ u_\varepsilon^s = B + \varepsilon (\partial_t u_\varepsilon^d)^s, \\ \varepsilon \partial_t u_\varepsilon^d(T) = 0, \\ u_\varepsilon(0) = u_0. \end{cases}$$

Setting $v_\varepsilon := \partial_t u_\varepsilon^d$, this equivalent to a forward-backward stochastic system:

$$\begin{cases} du_\varepsilon = v_\varepsilon dt + (B + \varepsilon G_\varepsilon) dW, \\ u_\varepsilon(0) = u_0, \end{cases} \quad \begin{cases} -\varepsilon dv_\varepsilon + v_\varepsilon dt + \partial \phi(u_\varepsilon) dt \ni -\varepsilon G_\varepsilon dW, \\ \varepsilon v_\varepsilon(T) = 0. \end{cases}$$

Difficulties:

- well-posedness of forward-backward system (ε fixed): backward stochastic equation
- equivalence of the Euler-Lagrange system (ε fixed): martingale representation theorems
- passage to the limit ($\varepsilon \rightarrow 0^+$): requires a forward-backward version of Itô's formula

The Euler–Lagrange equation in the stochastic case

Regularized equation at $\varepsilon > 0$:

$$\begin{cases} -\varepsilon \partial_t (\partial_t u_\varepsilon^d)^d + \partial_t u_\varepsilon^d + \partial \phi(u_\varepsilon) \ni 0, \\ u_\varepsilon^s = B + \varepsilon (\partial_t u_\varepsilon^d)^s, \\ \varepsilon \partial_t u_\varepsilon^d(T) = 0, \\ u_\varepsilon(0) = u_0. \end{cases}$$

Setting $v_\varepsilon := \partial_t u_\varepsilon^d$, this equivalent to a forward-backward stochastic system:

$$\begin{cases} du_\varepsilon = v_\varepsilon dt + (B + \varepsilon G_\varepsilon) dW, \\ u_\varepsilon(0) = u_0, \end{cases} \quad \begin{cases} -\varepsilon dv_\varepsilon + v_\varepsilon dt + \partial \phi(u_\varepsilon) dt \ni -\varepsilon G_\varepsilon dW, \\ \varepsilon v_\varepsilon(T) = 0. \end{cases}$$

Difficulties:

- well-posedness of forward-backward system (ε fixed): backward stochastic equation
- equivalence of the Euler-Lagrange system (ε fixed): martingale representation theorems
- passage to the limit ($\varepsilon \rightarrow 0^+$): requires a forward-backward version of Itô's formula

The Euler–Lagrange equation in the stochastic case

Regularized equation at $\varepsilon > 0$:

$$\begin{cases} -\varepsilon \partial_t (\partial_t u_\varepsilon^d)^d + \partial_t u_\varepsilon^d + \partial \phi(u_\varepsilon) \ni 0, \\ u_\varepsilon^s = B + \varepsilon (\partial_t u_\varepsilon^d)^s, \\ \varepsilon \partial_t u_\varepsilon^d(T) = 0, \\ u_\varepsilon(0) = u_0. \end{cases}$$

Setting $v_\varepsilon := \partial_t u_\varepsilon^d$, this equivalent to a forward-backward stochastic system:

$$\begin{cases} du_\varepsilon = v_\varepsilon dt + (B + \varepsilon G_\varepsilon) dW, \\ u_\varepsilon(0) = u_0, \end{cases} \quad \begin{cases} -\varepsilon dv_\varepsilon + v_\varepsilon dt + \partial \phi(u_\varepsilon) dt \ni -\varepsilon G_\varepsilon dW, \\ \varepsilon v_\varepsilon(T) = 0. \end{cases}$$

Difficulties:

- well-posedness of forward-backward system (ε fixed): backward stochastic equation
- equivalence of the Euler-Lagrange system (ε fixed): martingale representation theorems
- passage to the limit ($\varepsilon \rightarrow 0^+$): requires a forward-backward version of Itô's formula

Main results

Theorem (S.–Stefanelli, Comm. in PDEs (2021))

In the current setting, the following holds.

- For all $\varepsilon > 0$ the functional I_ε admits a unique global minimiser u_ε .
- The minimiser u_ε is the unique solution of the forward-backward Euler-Lagrange system, in the sense that there exists a triplet $(v_\varepsilon, \xi_\varepsilon, G_\varepsilon)$ such that

$$u_\varepsilon \in L^2(\Omega; C^0([0, T]; H)) \cap L^p(\Omega; L^p(0, T; V)),$$

$$v_\varepsilon \in L^2(\Omega; C^0([0, T]; V^*)) \cap L^2(\Omega; L^2(0, T; H)),$$

$$\xi_\varepsilon \in L^{p'}(\Omega; L^{p'}(0, T; V^*)),$$

$$G_\varepsilon \in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H))),$$

$$u_\varepsilon(t) = u_0 + \int_0^t v_\varepsilon(s) ds + \int_0^t (B(s) + \varepsilon G_\varepsilon(s)) dW(s) \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.},$$

$$\varepsilon v_\varepsilon(t) + \int_t^T v_\varepsilon(s) ds + \int_t^T \xi_\varepsilon(s) ds = -\varepsilon \int_t^T G_\varepsilon(s) dW(s) \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.},$$

$$\xi_\varepsilon(t) \in \partial\phi(u_\varepsilon(t)) \quad \text{for a.e. } t \in (0, T), \quad \mathbb{P}\text{-a.s.}$$

Main results

Theorem (S.–Stefanelli, Comm. in PDEs (2021))

As $\varepsilon \rightarrow 0^+$ it holds, for all $s \in (0, 1/2)$, that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u && \text{in } L^p(\Omega; L^p(0, T; V)) \cap L^{p'}(\Omega; W^{s,p'}(0, T; V_0^*)), \\ \xi_\varepsilon &\rightharpoonup \xi && \text{in } L^{p'}(\Omega; L^{p'}(0, T; V^*)), \\ -v_\varepsilon &\rightharpoonup \xi && \text{in } L^{p'}(\Omega; L^{p'}(0, T; V_0^*)), \\ \varepsilon v_\varepsilon &\rightarrow 0 && \text{in } L^{p'}(\Omega; C^0([0, T]; V_0^*)) \cap L^2(\Omega; L^2(0, T; H)), \\ \varepsilon G_\varepsilon &\rightarrow 0 && \text{in } L^{p'}(\Omega; L^2(0, T; \mathcal{L}^2(U, V_0^*))), \end{aligned}$$

where

$$\begin{aligned} u(t) + \int_0^t \xi(s) ds &= u_0 + \int_0^t B(s) dW(s) && \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.}, \\ \xi(t) &\in \partial\phi(u(t)) && \text{for a.e. } t \in (0, T), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

In particular, if $p < 4$ and $V \hookrightarrow H$ compactly, then

$$u_\varepsilon \rightarrow u \quad \text{in } L^r(\Omega; L^p(0, T; H)) \quad \forall r \in [1, p).$$

Sketch of the proof

Point 1: replace ϕ with the Moreau-Yosida regularisation ϕ_λ , $\lambda > 0$.

- define the regularized WED functionals

$$I_{\varepsilon,\lambda}(v) := \mathbb{E} \int_0^T e^{-t/\varepsilon} \left[\frac{\varepsilon}{2} \|\partial_t v^d(t)\|_H^2 + \phi_\lambda(v(t)) + \frac{1}{2} \|(v^s - B)(t)\|_{\mathcal{L}^2}^2 \right] dt$$

$$\text{if } v \in \mathcal{I}^{2,2}(H, H), \quad v(0) = u_0,$$

$$I_{\varepsilon,\lambda}(v) := +\infty$$

$$\text{if } v \in \mathcal{I}^{2,2}(H, H), \quad v(0) \neq u_0.$$

where

$$\mathcal{I}^{2,2}(H, H) := \left\{ v = v^d + v^s \cdot W : \begin{aligned} v^d &\in L^2(\Omega; H^1(0, T; H)), \\ v^s &\in L^2(\Omega; L^2(0, T; \mathcal{L}^2(U, H))) \end{aligned} \right\}.$$

- consider the approximated forward-backward system

$$\begin{cases} du_{\varepsilon,\lambda} = v_{\varepsilon,\lambda} dt + (B + \varepsilon G_{\varepsilon,\lambda}) dW, & \begin{cases} -\varepsilon dv_{\varepsilon,\lambda} + v_{\varepsilon,\lambda} dt + D\phi_\lambda(u_{\varepsilon,\lambda}) dt = -\varepsilon G_{\varepsilon,\lambda} dW, \\ \varepsilon v_{\varepsilon,\lambda}(T) = 0. \end{cases} \\ u_{\varepsilon,\lambda}(0) = u_0, \end{cases}$$

Sketch of the proof

Point 2: solve the problem at λ fixed and let $\lambda \rightarrow 0$

- $I_{\varepsilon,\lambda} : \mathcal{I}^{2,2}(H, H) \rightarrow [0, +\infty]$ admits a unique minimiser $u_{\varepsilon,\lambda}$
- characterise the subdifferential $\partial I_{\varepsilon,\lambda} : \mathcal{I}^{2,2}(H, H) \rightarrow 2\mathcal{I}^{2,2}(H, H)$
- $u_{\varepsilon,\lambda}$ is the unique solution to the regularised forward-backward problem

Point 3: let $\lambda \rightarrow 0$

- it holds that $u_{\varepsilon,\lambda} \rightarrow u_\varepsilon$ for some u_ε , in suitable topologies
- u_ε is the unique solution of the non-regularised forward-backward problem
- u_ε is the unique minimiser of the non-regularised WED functional I_ε

Point 4: let $\varepsilon \rightarrow 0$

- a forward-backward Itô formula holds
- $u_\varepsilon \rightarrow u$, where u is the unique solution of the original problem

Sketch of the proof

Point 2: solve the problem at λ fixed and let $\lambda \rightarrow 0$

- $I_{\varepsilon,\lambda} : \mathcal{I}^{2,2}(H, H) \rightarrow [0, +\infty]$ admits a unique minimiser $u_{\varepsilon,\lambda}$
- characterise the subdifferential $\partial I_{\varepsilon,\lambda} : \mathcal{I}^{2,2}(H, H) \rightarrow 2\mathcal{I}^{2,2}(H, H)$
- $u_{\varepsilon,\lambda}$ is the unique solution to the regularised forward-backward problem

Point 3: let $\lambda \rightarrow 0$

- it holds that $u_{\varepsilon,\lambda} \rightarrow u_\varepsilon$ for some u_ε , in suitable topologies
- u_ε is the unique solution of the non-regularised forward-backward problem
- u_ε is the unique minimiser of the non-regularised WED functional I_ε

Point 4: let $\varepsilon \rightarrow 0$

- a forward-backward Itô formula holds
- $u_\varepsilon \rightarrow u$, where u is the unique solution of the original problem

Sketch of the proof

Point 2: solve the problem at λ fixed and let $\lambda \rightarrow 0$

- $I_{\varepsilon, \lambda} : \mathcal{I}^{2,2}(H, H) \rightarrow [0, +\infty]$ admits a unique minimiser $u_{\varepsilon, \lambda}$
- characterise the subdifferential $\partial I_{\varepsilon, \lambda} : \mathcal{I}^{2,2}(H, H) \rightarrow 2\mathcal{I}^{2,2}(H, H)$
- $u_{\varepsilon, \lambda}$ is the unique solution to the regularised forward-backward problem

Point 3: let $\lambda \rightarrow 0$

- it holds that $u_{\varepsilon, \lambda} \rightarrow u_{\varepsilon}$ for some u_{ε} , in suitable topologies
- u_{ε} is the unique solution of the non-regularised forward-backward problem
- u_{ε} is the unique minimiser of the non-regularised WED functional I_{ε}

Point 4: let $\varepsilon \rightarrow 0$

- a forward-backward Itô formula holds
- $u_{\varepsilon} \rightarrow u$, where u is the unique solution of the original problem

Future research directions

WED approach for ill-posed stochastic evolution equations:

- non-uniqueness: selection of a solution (!)
- non-existence: generalised concept of solution (!!)
- extension of calculus of variations to stochastic framework

EXAMPLES

Doubly nonlinear stochastic evolution equations (Di Benedetto-Showalter):

$$\begin{cases} \partial_t(A(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} d(A(u)) + B(u) dt \ni G(u) dW, \\ u(0) = u_0. \end{cases}$$

Doubly nonlinear stochastic evolution equations (Colli-Visintin):

$$\begin{cases} A(\partial_t(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} u = u^d + u^s \cdot W, \\ A(\partial_t u^d) + B(u) \ni f, \\ u^s = G(u), \\ u(0) = u_0. \end{cases}$$

Future research directions

WED approach for ill-posed stochastic evolution equations:

- non-uniqueness: selection of a solution (!)
- non-existence: generalised concept of solution (!!)
- extension of calculus of variations to stochastic framework

EXAMPLES

Doubly nonlinear stochastic evolution equations (Di Benedetto-Showalter):

$$\begin{cases} \partial_t(A(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} d(A(u)) + B(u) dt \ni G(u) dW, \\ u(0) = u_0. \end{cases}$$

Doubly nonlinear stochastic evolution equations (Colli-Visintin):

$$\begin{cases} A(\partial_t(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} u = u^d + u^s \cdot W, \\ A(\partial_t u^d) + B(u) \ni f, \\ u^s = G(u), \\ u(0) = u_0. \end{cases}$$

Future research directions

WED approach for ill-posed stochastic evolution equations:

- non-uniqueness: selection of a solution (!)
- non-existence: generalised concept of solution (!!)
- extension of calculus of variations to stochastic framework

EXAMPLES

Doubly nonlinear stochastic evolution equations (Di Benedetto-Showalter):

$$\begin{cases} \partial_t(A(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} d(A(u)) + B(u) dt \ni G(u) dW, \\ u(0) = u_0. \end{cases}$$

Doubly nonlinear stochastic evolution equations (Colli-Visintin):

$$\begin{cases} A(\partial_t(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} u = u^d + u^s \cdot W, \\ A(\partial_t u^d) + B(u) \ni f, \\ u^s = G(u), \\ u(0) = u_0. \end{cases}$$

Future research directions

WED approach for ill-posed stochastic evolution equations:

- non-uniqueness: selection of a solution (!)
- non-existence: generalised concept of solution (!!)
- extension of calculus of variations to stochastic framework

EXAMPLES

Doubly nonlinear stochastic evolution equations (Di Benedetto-Showalter):

$$\begin{cases} \partial_t(A(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} d(A(u)) + B(u) dt \ni G(u) dW, \\ u(0) = u_0. \end{cases}$$

Doubly nonlinear stochastic evolution equations (Colli-Visintin):

$$\begin{cases} A(\partial_t(u)) + B(u) \ni f, \\ u(0) = u_0, \end{cases} \quad \rightsquigarrow \quad \begin{cases} u = u^d + u^s \cdot W, \\ A(\partial_t u^d) + B(u) \ni f, \\ u^s = G(u), \\ u(0) = u_0. \end{cases}$$

THANK YOU FOR YOUR ATTENTION!

<https://mat.univie.ac.at/~scarpa/>