Compactness of Sobolev embeddings with upper Ahlfors regular measures

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joint work with Paola Cavaliere (University of Salerno, Italy)

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*P. Cavaliere and Z. Mihula. Compactness of Sobolev-type embeddings with measures. *Commun. Contemp. Math.* 2021, Online First

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To investigate when a Sobolev embedding

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Definition (d-dimensional measure on $\overline{\Omega}$)

We say that a finite (nonnegative) Borel measure ν on $\overline{\Omega}$ is a d-dimensional measure on $\overline{\Omega}$ if it satisfies*

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{\nu\left(B(x,r)\cap\overline{\Omega}\right)}{r^d} < \infty \tag{1}$$

and there are a point $x_0 \in \overline{\Omega}$ and $r_0 > 0$ such that

$$\inf_{x\in\mathbb{R}^n, r\in(0,r_0)} \frac{\nu\left(B(x_0,r)\cap\overline{\Omega}\right)}{r^d} > 0.$$

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- λ_n, $\mathcal{H}^{n-1}|_{\partial\Omega}$, $\mathcal{H}^d|_{\Omega_d}$, $\mathrm{d}\nu(x) = |x-y_0|^{d-n}\,\mathrm{d}x$ ($y_0 \in \overline{\Omega}$ fixed)

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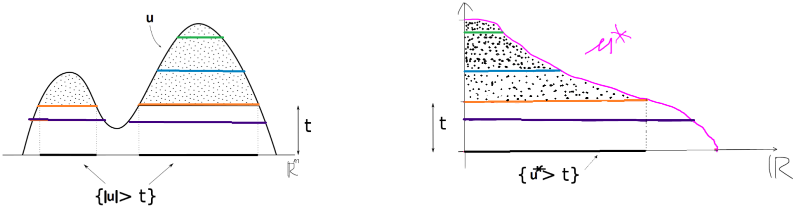
- $\|u\|_{X(R,\mu)} = \|u^*\|_{X(0,\mu(R))}$, where u^* is the nonincreasing rearrangement of u.



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- Do functions* from $W^m X(\Omega)$ leave meaningful traces on $\operatorname{supp} \nu$, i.e., is $u \in W^m X(\Omega)$ a well-defined function* in $L^1(\overline{\Omega}, \nu)$?

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- If $d \in [n-m, n]$, there is always a linear trace operator‡ $\mathrm{Tr}\colon W^m X(\Omega) \to L^{\frac{d}{n-m},1}(\overline{\Omega}, \nu)$.†

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- The r.i. space $X_{d,m,n}(\overline{\Omega}, \nu)$ whose associate space is defined by

$$\|u\|_{X'_{d,m,n}(\overline{\Omega},\nu)} = \left\| t^{-1+\frac{m}{n}} \int_0^{t^{\frac{d}{n}}} u^*(\nu(\overline{\Omega})s)\,\mathrm{d}s \right\|_{X'(0,1)}$$

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A. Cianchi, L. Pick, and L. Slavíková.
Sobolev embeddings, rearrangement-invariant spaces and Frostman measures. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(1):105–144, 2020.

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Theorem

Assume that $d \in [n-m, n]$. *Assume that** $Y \neq L^\infty$. *The following two statements are equivalent.*

1. $W^m X(\Omega) \hookrightarrow Y(\overline{\Omega}, \nu)$ *is compact.*
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- $L^{p,q} \overset{*}{\hookrightarrow} L^{r,s}$ iff $p > r$
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Assume that $d \in [n-m, n]$*. Assume that* $X \neq L^1$*. The following two statements are equivalent.*

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2. $X(\Omega) \overset{*}{\hookrightarrow} Y_{d,m,n}(\Omega)$*, where* $Y_{d,m,n}(\Omega)$ *is the optimal domain space in* $W^m X(\Omega) \hookrightarrow Y(\overline{\Omega}, \nu)$ *for* Y*.*

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- $W^m Y_{d,m,n}(\Omega) \hookrightarrow Y(\overline{\Omega}, \nu)$ is never compact; neither is $W^m \Lambda_{Y_{d,m,n}}(\Omega) \hookrightarrow Y(\overline{\Omega}, \nu)$.*

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- If $d \in (n-m, n]$ and $p \in [1, \frac{d}{n-m})$, $W^m L^1(\Omega) \hookrightarrow L^{p,1}(\overline{\Omega}, \nu)$ is compact, but $W^m L^1(\Omega) \hookrightarrow L^{\frac{d}{n-m},\infty}(\overline{\Omega}, \nu)$ is not.*

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- If $p_1 \in (1, \frac{n}{m})$ and $p_2 \in [1, \frac{dp_1}{n-mp_1})$, $W^m L^{p_1,\infty}(\Omega) \hookrightarrow L^{p_2,1}(\overline{\Omega}, \nu)$ is compact, but $W^m L^{p_1,1}(\Omega) \hookrightarrow L^{\frac{dp_1}{n-mp_1},\infty}(\overline{\Omega}, \nu)$ is not.

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- $W^m L^{\frac{n}{m},\infty}(\Omega) \hookrightarrow L^{p_2,1}(\overline{\Omega}, \nu)$ is compact for every $p_2 \in [1, \infty)$, but $W^m L^{\frac{n}{m},1}(\Omega) \hookrightarrow L^{\infty}(\overline{\Omega}, \nu)$ is not compact.

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- $W^m L^{\frac{n}{m},\infty}(\Omega) \hookrightarrow L^{p_2,1}(\overline{\Omega}, \nu)$ is compact for every $p_2 \in [1, \infty)$, but $W^m L^{\frac{n}{m},1}(\Omega) \hookrightarrow L^{\infty}(\overline{\Omega}, \nu)$ is not compact.
- $W^m L^{\frac{n}{m}}(\Omega) \hookrightarrow \exp L^{\alpha}(\overline{\Omega}, \nu)$ is compact iff $\alpha > \frac{n}{n-m}$.

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- $W^m L^{\frac{n-d}{m}}(\log L)^{\alpha}(\Omega) \hookrightarrow L^{\frac{n-d}{m}}(\overline{\Omega}, \nu)$ is compact if $\alpha > \frac{n-d}{m}$.

Thank you.