Spanning bipartite subgraphs of triangulations of a surface

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Our papers
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[2] Extension to even triangulations,

[3] Spanning bipartite quadrangulations of even triangulations,
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[4] Extension to 3-colorable triangulations,
Surfaces:
compact connected 2-manifolds without boundary

\( \mathcal{S}_k \): orientable surface of genus \( k \)

\( \mathcal{N}_k \): nonorientable surface of crosscap number \( k \)
Graphs on surfaces

Triangulation (tri.)

\( T \)

Quadrangulation (quad.)

\( Q \)
Graphs on surfaces

Eulerian (even) triangulation

Every vertex has even degree

Quadrangulation
For a given quad., we can extend it to a triangulation by adding a diagonal in each face.
Quad. ↔ Tri.

For a given triangulation, we often find a quad. as a spanning subgraph.
Additional requirements

For a given quad., can we extend it to

• Eulerian tri.?
• 3-colorable tri.?
• 4-connected tri.?

For a given tri., does it have

• bipartite quad.?
• 3-connected quad.?
Additional requirements

For a given quad., can we extend it to

[2] • Eulerian tri.? Yes
[4] • 3-colorable tri.? \( \exists \) iff condition
    • 4-connected tri.? Yes if it is simple

For a given tri., does it have

[3] • bipartite quad.? \( \exists \) iff condition for toroidal Eulerian tri.
    • 3-connected quad.? Yes if it is 5-connected
A spanning quad. subgraph

\[ T \]
Proposition A
Let $T$ be a loopless triangulation on a surface. Then $T$ has a spanning quad.
Proof
Problem

Let $T$ be a loopless triangulation of a surface. Does $T$ have a spanning bipartite quad.?

Remark

- Every plane quad. is bipartite.
- 4-colorability of $T$ is a sufficient condition.
Proposition B
If $T$ is a 4-colorable tri. on a surface, then $T$ has a spanning bipartite quad.
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On the projective plane

Theorem 1 (Kündgen, Thomassen, 2017; Nakamoto, N., Ozeki, 2019)
Let $T$ be an Eulerian tri. of the projective plane.
If $T$ is 3-colorable, then every spanning quad. of $T$ is bipartite. If $T$ is not 3-colorable, then $T$ has both bipartite and non-bipartite spanning quads.
On the torus

Fact
Even tri. $K_7$ of the torus has no spanning bipartite quad.
A known theorem

Theorem C (Kündgen, Thomassen, 2017)

Let $T$ be a loopless Eulerian tri. of the torus. Then $T$ has a spanning non-bipartite quad. Furthermore, if $T$ has sufficiently large edge width, then $T$ has a spanning bipartite quad.
Main theorem

Theorem 2 (Nakamoto, N., Ozeki, 2019)
Let $T$ be a loopless Eulerian tri. of the torus. $T$ has a spanning bipartite quad. if and only if $T$ does not have $K_7$ as a subgraph.
We use a “generating” theorem.

**Theorem D** (Matsumoto, Nakamoto, Yamaguchi, 2018)
Every loopless Eulerian tri. of the torus is generated from one of 27 minimal graphs and 6-regular tris. by using 4-splittings and 2-vertex additions.
Generating Eulerian tris.

Eulerian tris. on the torus

$|V(T)|$

$K_7$

: Minimal graph

: 4-splitting

: 2-vertex addition
27 minimal graphs
Two operations for the generating theorem

4-contraction

4-splitting

2-vertex removal

2-vertex addition
2-vertex addition

\[ T' \]

\[ T'' \]

\[ T \]
4-splitting

$T'$

$T''$

$T$
Outline of the Proof of Thm 2

We use the generating theorem.

(i) Confirming that all minimal graphs other than $K_7$ have a spanning bipartite quad.

(ii) Showing that the bipartiteness of a spanning quad. is preserved under the two operations.

Theorem 2 (Nakamoto, N., Ozeki, 2019)

Let $T$ be a loopless Eulerian tri. of the torus. $T$ has a spanning bipartite quad. if and only if $T$ does not have $K_7$ as a subgraph.
Spanning bipartite quads.
Conclusion

For a given quad., can we extend it to

[2] • Eulerian tri.? Yes

[4] • 3-colorable tri.? \( \exists \) iff condition
• 4-connected tri.? Yes if it is simple

For a given tri., does it have

[3] • bipartite quad.? \( \exists \) iff condition for toroidal Eulerian tri.
• 3-connected quad.? Yes if it is 5-connected
Thank you!