Dynamics of visco-elastic bodies with a cohesive interface

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based on joint works with R. Scala

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Bulk energies

\[ \mathcal{E}(u) = \frac{1}{2} \int_{\Omega} \mu |\nabla u|^2 \, dx \quad \mathcal{K}(v) = \frac{1}{2} \int_{\Omega} \rho |v|^2 \, dx \quad \mathcal{R}(v) = \frac{1}{2} \int_{\Omega} \eta |\nabla v|^2 \, dx \]

Antiplane displacement \( u \)

Internal (history) variable "\( \xi(t) = \max_{[0,t]} [\|u\|] \)" on the interface (crack path) \( K \)
Karush-Kuhn-Tucker conditions

\[ \dot{\xi}(t) \geq 0 \quad \text{and} \quad \dot{\xi}(t)(\xi(t) - \|u(t)\|) = 0 \quad \text{and} \quad \|u(t)\| \leq \xi(t) \]

horizontal (reversible) + diagonal (irreversible) directions inside the cone

\[ [u] = w \]
Cohesive energy: loading and unloading

\[ \hat{\psi}(w) = \int K \psi(|u|, \xi) \, dr \]

\[ \hat{\psi}'(w) = \int K \psi'(|u|, \xi) \, dr \]

\[ \psi(w, \xi) = \begin{cases} \hat{\psi}(|w|) & \text{if } |w| \geq \xi \\ \hat{\psi}(\xi) - \frac{1}{2} \left( \frac{\hat{\psi}'(\xi)}{\xi} \right) (\xi^2 - w^2) & \text{if } |w| < \xi \end{cases} \]

(keywords: finite tension, softening)

[Dagdale, Barenblatt, ..., Ortiz-Pandolfi]

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fracture greatly extends what should be the limits of its applicability to those comparatively rare materials that are perfectly brittle. Experimental investigations show that when cracks appear some materials, which behave as highly plastic bodies in common tensile tests, fracture in such a way that plastic deformations, though present, are concentrated in a thin layer near the crack surface.

D. K. Felbeck and E. O. Orowan [28] carried out experiments on fracture of low-carbon steel plates with a saw-cut crack under conditions corresponding to Griffith’s scheme of uniform extension. Experimental results are in good agreement with Griffith’s formula, but the surface-energy density exceeds by about three orders of magnitude the surface tension of the material investigated. It seems plausible to assume that within the specified order of

\[ d = \xi_c \text{ "cohesive zone"} \]

\[ \xi_c \to 0^+ \text{ brittle fracture (Griffith)} \]

**Issue:** brittle fracture is incompatible with visco-elasticity (in q.s.)
Dissipated energy

For $\psi_d(\xi) = \psi(0, \xi)$

$$\Psi(u, \xi) = \int_K \psi_s(\|u\|, \xi) + \psi_d(\xi) \, dr = \Psi_s(u, \xi) + \Psi_d(\xi)$$

$\psi_d$ is monotone and bounded

Potential energy $\mathcal{F}(t, u, \xi) = \mathcal{E}(u) + \Psi(u, \xi) - \langle f(t), u \rangle$
Weak solutions

Let \( U = \{ u \in H^1 : u = 0 \text{ on } \partial D \Omega \} \) with \( H^1 \)-norm.

Weak solutions in \( W^{1,2}(0, T; U) \cap W^{2,2}(0, T; U^*) \)

System of PDEs

\[
\begin{cases}
\rho \ddot{u}(t) + \partial_u F(t, u(t), \xi(t)) + \partial_v R(\dot{u}) \geq 0 \\
\dot{\xi}(t) \geq 0 \text{ and } \dot{\xi}(t)(\xi(t) - \|u(t)\|) = 0 \text{ and } \|u(t)\| \leq \xi(t) \\
u(0) = u_0, \ \xi(0) = \xi_0, \ \dot{u}(0) = v_0
\end{cases}
\]

\((\rho \ddot{u}(t), \phi)_U + \partial_u F(t, u(t), \xi(t); \phi) + \partial_v R(\dot{u}(t))[\phi] \geq 0 \text{ for } \phi \in U\)

\[
\partial_u \Psi(u, \xi; \phi) = \int_K \partial_w \psi([u], \xi; \phi) \, dr
\]
Energy identity

For every time $t$

$$\mathcal{F}(t, u(t), \xi(t)) + \mathcal{K}(\dot{u}(t)) = \mathcal{F}(0, u_0, \xi_0) + \mathcal{K}(v_0)$$

$$+ \int_0^t \partial_t \mathcal{F}(s, u(s), \xi(s)) \, ds - \int_0^t \partial_v \mathcal{R}(\dot{u}(s)) [\dot{u}(s)] \, ds$$

$$\mathcal{E}(u(t)) + \Psi_s(u(t), \xi(t)) + \mathcal{K}(\dot{u}(t)) = \mathcal{E}(u_0) + \Psi_s(u_0, \xi_0) + \mathcal{K}(v_0)$$

$$+ \int_0^t \mathcal{P}_{\text{ext}}(s, u(s)) \, ds$$

$$- \int_0^t \partial_v \mathcal{R}(\dot{u}(s)) [\dot{u}(s)] \, ds - \int_0^t \partial_\xi \Psi_d(\xi(s); \dot{\xi}(s)) \, ds$$
Adhesive energy and dissipation: $0 \leq z \leq 1$ and $\dot{z} \leq 0$

\[ \Phi_s(u, z) = \int_K z \|u\|^2 \, dr \quad \Phi_d(z) = \int_K 1 - z \, dr \]

\[ \Psi_s(u, \xi) = \int_K \left( \frac{\hat{\psi}'(\xi)}{\xi} \right) \|u\|^2 \, dr \quad \Psi_d(\xi) = \int_K \psi_d(\xi) \, dr \]

\[
\begin{cases}
\rho \ddot{u}(t) + \partial_u F(t, u(t), \xi(t)) + \partial_v R(\dot{u}) \geq 0 \\
\xi(t) \in \text{argmin} \{ F(t, u(t), \zeta) : \zeta \geq \xi(t) \} \\
F(t, u(t), \xi(t)) + K(\dot{u}(t)) \leq \mathcal{F}(0, u_0, \xi_0) + K(v_0)
\end{cases}
\]

\[
+ \int_0^t \partial_t F(s, u(s), \xi(s)) \, ds - \int_0^t \partial_v R(\dot{u}(s)) [\dot{u}(s)] \, ds
\]
For \( \sigma(t) = \mu \nabla u(t) + \nu \nabla \dot{u}(t) \) the visco-elastic stress

Solutions in \( W^{1,\infty}(0, T; U) \cap W^{2,\infty}(0, T; L^2) \) solve

\[
\begin{cases}
\rho \ddot{u}(t) = \text{div} \sigma(t) + f(t) & \text{in } \Omega \\
\sigma(t) \nu = 0 & \text{in } \partial_N \Omega \\
\sigma^+(t) \nu = \sigma^-(t) \nu \in \partial_w \psi([u(t)], \xi(t)) & \text{in } K
\end{cases}
\]

Compatibility assumptions on the initial conditions:

- \( v_0 \in U \) with \([v_0] = 0\) if \( \xi_0 = 0\) and \( \int_{\xi_0 > 0} \left( \frac{\hat{\psi}'(\xi_0)}{\xi_0} \right) |[v_0]|^2 \, dr < \infty \)
  (e.g. if there exists \( \tau_0 > 0\) s.t. \(|[u_0 + \tau_0 v_0]| \leq \xi_0\))

- there exists \( w_0 \in L^2 \) s.t. \( \rho w_0 + \partial_u \mathcal{F}(0, u_0, \xi_0) + \partial_v \mathcal{R}(v_0) \geq 0 \)
Existence of weak solutions

1) Replace $\xi_0$ with $\xi_\varepsilon = \max\{\xi_0, \varepsilon\} \Rightarrow "adhesive"$ regularization of $\Psi$

2) Time discretization $t_{n,k} = k\tau_n$ for $\tau_n = T/n$

\[
\begin{align*}
\begin{cases}
  u_{n,k} \in \text{argmin} \{ J(t_{n,k}, u, \xi_{n,k-1}) : u \in \mathcal{U} \} \\
  \xi_{n,k} = \max\{\xi_{n,k-1}, ||u_{n,k}||\}
\end{cases}
\end{align*}
\]

\[
J = \frac{1}{2} \rho \left\| \frac{u - 2u_{n,k-1} + u_{n,k-2}}{\tau_n} \right\|^2 + R \left( \frac{u - u_{n,k-1}}{\tau_n} \right) + F(t_{n,k}, u, \xi_{n,k-1})
\]

\[
\begin{align*}
\begin{cases}
  u_{n,k} \in \text{argmin} \{ J(t_{n,k}, u, \xi_{n,k}) : u \in \mathcal{U} \} \\
  \xi_{n,k} \in \text{argmin} \{ J(t_{n,k}, u_{n,k}, \xi) : \xi \geq \xi_{n,k-1} \}
\end{cases}
\end{align*}
\]
Existence of weak solutions

1) Replace $\xi_0$ with $\xi_\varepsilon = \max\{\xi_0, \varepsilon\} \Rightarrow$ “adhesive” regularization of $\Psi$

2) Time discretization $t_{n,k} = k\tau_n$ for $\tau_n = T/n$

$$
\left\{
\begin{array}{l}
u_{n,k} \in \text{argmin} \left\{ J(t_{n,k}, u, \xi_{n,k-1}) : u \in U \right\} \\
\xi_{n,k} = \max\{\xi_{n,k-1}, ||u_{n,k}||\}
\end{array}
\right.
$$

$$
J = \frac{1}{2} \rho \left\| \frac{u - 2u_{n,k-1} + u_{n,k-2}}{\tau_n} \right\|^2 + R \left( \frac{u - u_{n,k-1}}{\tau_n} \right) + F(t_{n,k}, u, \xi_{n,k-1})
$$

$$
\left\{ \begin{array}{l}
\langle \rho \dot{v}_{n,k}, \phi \rangle + \partial_v R(u_{n,k})[\phi] + \partial_u F(t_{n,k}, u_{n,k}, \xi_{n,k})[\phi] = 0 \\
||u_{n,k}|| \leq \xi_{n,k} \quad \dot{\xi}_{n,k}(||u_{n,k}|| - \xi_{n,k}) = 0 \quad \dot{\xi}_{n,k} \geq 0
\end{array} \right.
$$

$\psi_d(\xi)$
Existence of weak solutions

3) Piecewise constant/affine interpolation and $\varepsilon$-uniform compactness ...

$$\|u_n\|_{W^{1,2}(0,T;U)} + \|\dot{u}_n\|_{W^{1,2}(0,T;U^*)} \leq C \quad \Rightarrow \quad \|\xi_n\|_{W^{1,2}(0,T;L^2)} \leq C$$

by the discrete E-L equation $\partial_u J(t_{n,k}, u_{n,k}, \xi_{n,k})[\phi]$ with $\phi = u_{n,k} - u_{n,k-1}$

4) Improved convergence $\xi_n \to \xi^\varepsilon$ in $L^2(0,T;L^2)$ by KKT

[\text{N.-Vitali}]

5) Euler-Lagrange eq. $\partial_u J(t_{n,k}, u_{n,k}, \xi_{n,k})[\phi] = 0$ converge to

$$\rho(\ddot{u}^\varepsilon(t), \phi) + \partial_u F(t, u^\varepsilon(t), \xi^\varepsilon(t))\phi + \partial_v R(\dot{u}^\varepsilon)[\phi] = 0$$

6) KKT from discrete to continuum ...

7) Convergence for $\varepsilon \to 0$ ...

$$\rho(\ddot{u}(t), \phi) + \partial_u F(t, u(t), \xi(t))\phi + \partial_v R(\dot{u})[\phi] \ni 0$$
1) \( t \mapsto K(\dot{u}(t)) + \mathcal{E}(u(t)) + \Psi(u(t), \xi(t)) \) is AC

\[
|\Psi(u(t), \xi(t)) - \Psi(u_0, \xi_0)| \leq C \int_0^t \|\dot{\mathcal{J}}(s)\| \, ds \quad \text{by KKT}
\]

2) \( \dot{K} + \dot{\mathcal{E}} + \dot{\Psi} = (\rho \ddot{u}(t), \dot{u}(t)) + \partial_u \mathcal{E}(u(t))[\dot{u}(t)] + \partial_u \Psi(u(t), \xi(t); \dot{u}(t)) \)

\[
\Psi(u(t), \xi(t)) - \Psi(u_0, \xi_0) = \int_0^t \partial_u \Psi(u(s), \xi(s); \dot{u}(s)) \, ds \quad \text{by KKT}
\]

3) E-L equation for \( \phi = \dot{u} \)

\[
(\rho \dddot{u}(t), \dddot{u}(t)) + \partial_u \mathcal{F}(t, u(t), \xi(t); \dot{u}(t)) + \partial_v \mathcal{R}(\dddot{u}(t))[\dot{u}(t)] = 0.
\]

equality relies on \( \|\dddot{u}(t)\| = 0 \) if \( \xi(t) = 0 \) \quad \text{by KKT}
Existence of strong solutions

1) Let $\tilde{\xi}_\varepsilon = \max\{\xi_0, \varepsilon\}$. Replace the i.c. $u_0$ and $\xi_0$ with

$$u_\varepsilon \in \text{argmin} \left\{ \mathcal{F}(0, u, \tilde{\xi}_\varepsilon) + \langle \eta \nabla v_0, \nabla u \rangle + \langle \rho w_0, u \rangle : u \in U \right\}$$

$$\xi_\varepsilon = \max\{\tilde{\xi}_\varepsilon, \|u_\varepsilon\|\}$$

Then $u_\varepsilon \to u_0$ and $\xi_\varepsilon \to \xi_0$ (by $\Gamma$-convergence)

the compatibility assumptions hold

2) Time discretization and incremental scheme ...

3) Interpolation and (tricky) $\varepsilon$-uniform compactness for the speed $v_n$

$$\|v_n\|_{L^\infty(0,T;U)} + \|\dot{v}_n\|_{L^\infty(0,T;L^2)} \leq C.$$

by studying the difference of the E-L equations

$$\left( \partial_u \mathcal{J}(t_{n,k}, u_{n,k}, \xi_{n,k}) - \partial_u \mathcal{J}(t_{n,k-1}, u_{n,k-1}, \xi_{n,k-1}) \right)[v_{n,k} - v_{n,k-1}] = 0$$
Existence of strong solutions

Linear (bulk) terms are easy but

\[
\partial_u \Psi(u_{n,k}, \xi_{n,k})[v_{n,k} - v_{n,k-1}] - \partial_u \Psi(u_{n,k-1}, \xi_{n,k-1})[v_{n,k} - v_{n,k-1}] = \int_K \tau_n \alpha_{n,k} [v_{n,k}][v_{n,k} - v_{n,k-1}] \, ds
\]

where \(\tau_n v_{n,k} = u_{n,k} - u_{n,k-1}\) and

\[
\alpha_{n,k} = \frac{c_{n,k} [u_{n,k}] - c_{n,k-1} [u_{n,k-1}]}{[u_{n,k} - u_{n,k-1}]} \quad \text{for} \quad c_{n,k} = \frac{\hat{\psi}'(\xi_{n,k})}{\xi_{n,k}}
\]

Use the visco-elastic term

\[
\| \nabla v_{n,k} - \nabla v_{n,k-1} \|^2_{L^2} \geq C \int_K [v_{n,k} - v_{n,k-1}]^2 \, ds
\]

After tricky estimates on \(\alpha_{n,k} - \alpha_{n,k-1}\) ... Gronwall Lemma gives compactness
Generalizations

Conditions on the energy density $\hat{\psi} : [0, +\infty) \to [0, +\infty)$ for weak solutions

- $\hat{\psi}(0) = 0$, $\hat{\psi}(w) > 0$ for $w > 0$, $\hat{\psi}(w) = \hat{\psi}(\xi_c)$ for $w \geq \xi_c > 0$
- $\hat{\psi}$ is concave
- $\hat{\psi}$ is of class $C^1$ in $[0, +\infty)$ and of class $C^2$ in $[0, \xi_c]$.

Similarly for $\xi_c = +\infty$ ...

A non-linear dissipation pseudo-potential  \( \psi_d(\xi) = \hat{\psi}(\xi) - \frac{1}{2} \hat{\psi}'(\xi)\xi \)

Further conditions for strong solutions

- $\hat{\psi}'$ is concave in $[0, \xi_c]$
- there exists \( c > 0 \) such that \( \mu \| \nabla u \|^2 + \beta \| [u] \|^2 \geq c \| \nabla u \|^2 \)
  where \( \beta = \min \{ \hat{\psi}''(w) : w \in [0, \xi_c] \} < 0 \)
Some open problems

Uniqueness ...

**Vectorial setting:** \( u \in H^1(\Omega; \mathbb{R}^N) \) and \( N = 2, 3 \)

\[
\Psi(u, \xi) = \int_K \psi([u], \xi) \, dr \quad \hat{\psi}(w) = \begin{cases} 
\ldots & \text{if } w \cdot \nu \geq 0 \\
+\infty & \text{otherwise}
\end{cases}
\]

Some results for the existence of weak solutions ...

[Scala, Scala-Schimperna]

**Quasi-static limit** by time rescaling: \( v_0 = 0 \) and \( \bar{f}_n(t) = f(t/n) \) for \( t \in [0, nT] \)

consider \( \bar{u}_n : [0, nT] \to U \) and \( u_n(t) = \bar{u}_n(nt) : [0, T] \to U \) solving

\[
\rho_n \dddot{u}_n(t) + \eta_n \partial_v \mathcal{R}(\dot{u}_n(t)) + \partial_u \mathcal{F}(t, u_n(t), \xi_n(t)) \ni 0
\]

where \( \rho_n = \rho n^{-2} \) and \( \eta_n = \eta n^{-1} \)

Issue: characterize as \( n \to \infty \) the limit q.s. evolution

[Scala, Roubíček, Scilla-Solombrino]
Open problem: which q.s. evolution?

Qualitatively: discontinuous evolutions $u \in BV(0, T; H^1)$ and $\xi \in BV(0, T; L^2)$

- $\partial_u F(t, u(t), \xi(t)) \ni 0$ and “KKT”
- characterization of the transition in discontinuities

$$[0, S] \ni s \mapsto \bar{u}(s) \text{ s.t. } \bar{u}(0) = u^-(t) \text{ and } \bar{u}(S) = u^+(t)$$

- energy identity

$$F(t, u(t), \xi(t)) = F(0, u_0, \xi_0) + \int_0^t \partial_t F(s, u(s), \xi(s)) \, ds + \sum_{t_j \in J} [F(t_j, u(t_j), \xi(t_j))] + \mu([0, t])$$  \[Scala, Roubíček\]
A q.s. evolution with no inertia

Balanced (or vanishing) viscosity evolution for $\rho = 0$ and $\eta_n \to 0$

Discontinuous evolutions $u \in BV(0, T; H^1)$ and $\xi \in BV(0, T; L^2)$

Parametrization $s \mapsto (t(s), u(s), \xi(s))$ [Mielke-Efendiev, N., N.-Vitali]

- if $t(s)$ is a continuity time

\[
\partial_u \mathcal{F}(t(s), u(s), \xi(s)) \ni 0
\]

\[
\xi'(s) \geq 0, \quad \xi'(s)(\xi(s) - \|u(s)\|) = 0, \quad \|u(s)\| \leq \xi(s)
\]

- if $[s_1, s_2]$ is a parametrization of a jump then $t = t(s)$ and

\[
\eta \partial_v \mathcal{R}(u'(s)) + \partial_u \mathcal{F}(t, u(s), \xi(s)) \ni 0
\]

\[
\eta u'_n(s) \approx \dot{u}_n(t_n(s)) \eta_n \quad \text{where} \quad \eta_n = \eta n^{-1}
\]