Laplacians on Infinite Graphs

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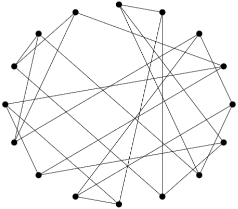
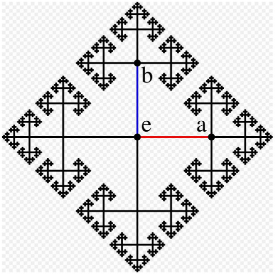
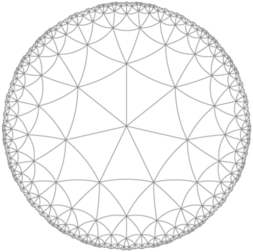
Graphs

Definition

A **graph** $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is a set of **vertices** $\mathcal{V}$ and **edges** $\mathcal{E}$.







Assumptions

- $\mathcal{V}$ and $\mathcal{E}$ are at most countable, and $\mathcal{G}_d$ is **connected**
- $\mathcal{G}_d$ is **locally finite** (vertex degree: $\deg(v) < \infty$, $v \in \mathcal{V}$)

Weighted Laplacians on Graphs

- The **combinatorial Laplacian**

$$(L_{\mathrm{comb}} f)(v) = \sum_{u\sim v} f(v) - f(u) = \deg(v) f(v) - \sum_{u\sim v} f(u).$$

$L_{\mathrm{comb}} \sim$ the **adjacency matrix** (Spectral Graph Theory; @8ECM:**MS-46**).

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- The **normalized Laplacian** (*physical Laplacian* or *Markov operator*)

$$(L_{\mathrm{norm}} f)(v) = \frac{1}{\deg(v)} \sum_{u \sim v} f(v) - f(u) = f(v) - \frac{1}{\deg(v)} \sum_{u \sim v} f(u).$$

L_{norm} generates a simple random walk on $\mathcal{G}_d$:

Definition

G a **finitely generated group**, S a finite **generating set**, $S = S^{-1}$.
The **Cayley graph** $C(\mathrm{G}, S)$ is the graph with $\mathcal{V} = \mathrm{G}$ and $x \sim y \Leftrightarrow x^{-1} y \in S$.

H. Kesten, *Symmetric random walks on groups*, Trans. AMS (1958).

WARNING: On Cayley graphs, $\deg \equiv \# S$ and hence $L_{\mathrm{comb}} = \# S \cdot L_{\mathrm{norm}}$.

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L_{norm} generates a simple random walk on $\mathcal{G}_d$.

- **Discrete-time Markov chain**: $b\colon \mathcal{E} \to \mathbb{R}_{>0}$, set $m_b(v) = \sum_{u\sim v} b(e_{u,v})$. Then

$$(L_{\mathrm{b}} f)(v) = \frac{1}{m_b(v)} \sum_{u\sim v} b(e_{u,v})(f(v) - f(u))$$

generates a discrete time random walk: $\mathrm{Prob}(X_{n+1} = u \,|\, X_n = v) = \frac{b(e_{u,v})}{\sum_{u\sim v} b(e_{u,v})}$.

Weighted Laplacians on Graphs

$(\mathcal{V}, m; b)$ with $m\colon \mathcal{V} \to (0,\infty)$ a **vertex weight**, and $b\colon \mathcal{V}\times\mathcal{V} \to [0,\infty)$ s.t.

- symmetric, $b(u,v) = b(v,u)$, and vanishing diagonal, $b(v,v) = 0$,
- locally finite: $\#\{u \,|\, b(v,u) > 0\} < \infty$,

is called **a weighted graph over $(\mathcal{V}, m)$.**

The (formal) **Laplacian** $L = L_{\mathcal{V},m,b}$ is

$$(Lf)(v) = \frac{1}{m(v)} \sum_{u\in\mathcal{V}} b(u,v)(f(v) - f(u)), \qquad v \in \mathcal{V}.$$

WARNING! **"formal"** since **L might be unbounded!**

- **Combinatorial Laplacian**:

  Take $L = L_{\mathrm{comb}}$, that is, $m \equiv 1$ on $\mathcal{V}$ and $b =$ adjacency matrix.

  L_{comb} is **bounded** exactly when $\mathcal{G}_d$ has **bounded geometry** ($\sup\deg < \infty$)

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Dirichlet forms on discrete measure spaces

In $\ell^2(\mathcal{V}; m)$, the **energy form** (at least on $f \in C_c(\mathcal{V})$)

$$\mathfrak{q}[f] = \langle Lf, f \rangle_{\ell^2(\mathcal{V};m)} = \tfrac{1}{2} \textstyle\sum_{u,v} b(u, v)|f(v) - f(u)|^2.$$

Dirichlet form is a closed symmetric **Markovian form** on an L^2 space:
Beurling–Deny conditions: $\mathfrak{q}[|f|] \le \mathfrak{q}[f]$ and $\mathfrak{q}[0 \vee f \wedge 1] \le \mathfrak{q}[f]$

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"**Dirichlet forms** on discrete measure spaces are **weighted graphs**"

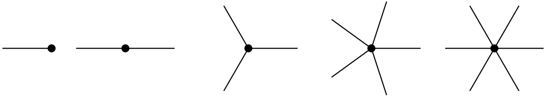
M. Keller and D. Lenz// Crelle's J. (2012).

Definition (a.k.a. "cable graphs" or "metrized graphs")

$\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is a connected, locally finite graph.
If every edge $e \in \mathcal{E}$ is assigned with a positive finite length $|e| \in (0, \infty)$, then $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$ is called a metric graph

Metric Graph as ...

- a simplicial 1-complex,
- a topological space, which looks locally like a star-graph



- a length space when equipped with a natural ("geodesic") path metric – a distance between two points is the arc-length of "shortest" path,
- a (real) 1D manifold with singularities: vertices of degree ≥ 3 are "branching" points; degree $= 1$ are "boundary" points,
- a non-Archimedean analog of Riemann surfaces

  a *tropical curve* or a degeneration of a smooth family of Riemann surfaces

Metric Graphs and Their Laplacians

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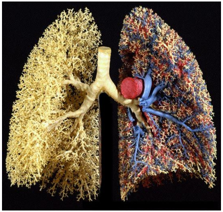
Definition

Quantum graphs are Laplacians on (weighted) metric graphs.

Applications: “thin wire materials” in physics/biology/...



lungs $\approx$ *binary tree of 20-23 generations*
approx. $2 \times 10^6 - 1.6 \times 10^7$ *vertices*

Cast of human lungs (photo by E. Weibel)

P. Joly, M. Kachanovska, and A. Semin, *Netw. Heterog. Media* (2019)

Metric Graphs and Their Laplacians

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Further applications:

- Quantum ergodicity (Anantharaman, Berkolaiko, Colin de Verdière, . . .)
- Counting spectral measures as 1D Fourier quasi-crystals (Kurasov–Sarnak'2020)
- . . . **@8ECM: MS-26, MS-29, MS-40, MS-48, . . .**

G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, Amer. Math. Soc., 2013

Laplacians on Metric Graphs

Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$, identify each edge $e \in \mathcal{E}$ with $\mathcal{I}_e = [0, |e|]$. Let $\mu, \nu \colon \mathcal{E} \to (0, \infty)$ be **edge weights**, $(\mathcal{G}, \mu, \nu)$ is a **weighted metric graph**.

$$L^2(\mathcal{G}; \mu) \cong \bigoplus_{e\in\mathcal{E}} L^2(e; \mu_e), \qquad \mu_e(\mathrm{d}x) := \mu_e \mathrm{d}x_e \text{ on } e = \mathcal{I}_e.$$

Kirchhoff Laplacian (weighted "Laplace–Beltrami" on $\mathcal{G}$)

Δ acts as $\frac{1}{\mu_e}\frac{\mathrm{d}}{\mathrm{d}x_e}\nu_e\frac{\mathrm{d}}{\mathrm{d}x_e}$ on the interior of $\mathcal{G}$, and boundary conditions:

Kirchhoff conditions: $\begin{cases} f \text{ is continuous at } v \\ \sum_{e\in\mathcal{E}_v} \nu_e \partial_e f(v) = 0 \end{cases}, \quad v \in \mathcal{V}.$

- $\deg(v) = 1$: Kirchhoff = Neumann at v, $\partial_e f(v) = 0$,
- $\deg(v) = 2$: Kirchhoff = continuity of f and its (weighted) derivative at v ("removable" singularity/inessential vertex)

Laplacians on Metric Graphs

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The **maximal Kirchhoff Laplacian** Δ_{Kir} is defined in $L^2(\mathcal{G}; \mu)$ on the domain

$$\mathrm{dom}(\Delta_{\mathrm{Kir}}) = \big\{ f \in H^2(\mathcal{G} \setminus \mathcal{V}) \,|\, \text{(Kirchhoff) on } \mathcal{V} \big\}.$$

The **minimal Kirchhoff Laplacian** $\Delta_{\mathrm{Kir},0}$ is the L^2 closure of

$$\Delta \restriction \mathrm{dom}(\Delta_{\mathrm{Kir}}) \cap L^2_c(\mathcal{G}).$$

Harmonic Functions of Graphs

f is harmonic on $(\mathcal{G},\mu,\nu)$ if $\Delta f = 0$ on $\mathcal{G}$, i.e., f is edgewise affine and Kirchhoff conditions. By continuity, f can be identified with $f|_{\mathcal{V}}$ and its slopes at $v\in\mathcal{V}$

$$\sum_{u\sim v} \nu_{e_{u,v}} \frac{f(u)-f(v)}{|e_{u,v}|} = 0.$$

Definition: $(\mathcal{G},\mu,\nu)$ has **finite intrinsic size** if $\sup_{e\in\mathcal{E}} |e|\sqrt{\frac{\mu_e}{\nu_e}} < \infty$.

Moreover, $f\in L^2(\mathcal{G};\mu)$ if and only if $f|_{\mathcal{V}} \in \ell^2(\mathcal{V};m)$, where

$$m(v) = \sum_{e\sim v} \mu_e |e|.$$

Define a graph Laplacian $L = L(\mathcal{G},\mu,\nu)$

$$(Lf)(v) = \frac{1}{m(v)} \sum_{u\sim v} \frac{\nu_{e_{u,v}}}{|e_{u,v}|}(f(u)-f(v))$$

L and Δ have the **same harmonic functions**

If $(\mathcal{G},\mu,\nu)$ has finite **intrinsic size**, then $\ker(\Delta_{\mathrm{Kir}}) \cong \ker(L(\mathcal{G},\mu,\nu))$

Connections between discrete and metric graphs

Theorem (discrete vs continuous)

The Laplacians Δ_{Kir} and $L = L(\mathcal{G}, \mu, \nu)$ **share many basic**

- **Spectral properties** (Exner-AK-Malamud-Neidhardt'18, AK–Nicolussi'21)
  - **Self-adjoint uniqueness** (N. Nicolussi @8ECM)
  - **Positive spectral gap**
  - **Ultracontractivity estimates**
  - ...
- **Parabolic properties**
  - **Markovian uniqueness** *(AK–Nicolussi'21)* (N. Nicolussi @8ECM)
  - **Recurrence/transience** *(Haeseler'14, AK–Nicolussi'21)*
  - **Stochastic completeness** *(Folz'14, ...,)* (X. Huang @8ECM)
  - ...

N. Varopoulos, *Long range estimates for Markov chains*, Bull. Sci. Math. (1985)

P. Exner, A. Kostenko, M. Malamud, H. Neidhardt, *Spectral theory of infinite quantum graphs*, Ann. Henri Poincaré (2018)

Analysis on weighted graphs

A lot of parallels between analysis on manifolds and analysis on graphs.

However, what is the **right choice of a metric** on a graph?

E.B. Davies, *Analysis on graphs and noncommutative geometry*, JFA (1993)

Combinatorial distance (a.k.a. *word metric* on groups) a lot of controversy!

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Definition (Frank, Lenz & Wingert, J. Funct. Anal. (2014))

A metric $\varrho\colon \mathcal{V}\times\mathcal{V}\to[0,\infty)$ is called **intrinsic** w.r.t. $(\mathcal{V}, m; b)$ if

$$\sum_{u\in\mathcal{V}} b(u,v)\varrho(u,v)^2 \le m(v), \qquad v\in\mathcal{V}.$$

Examples

$b =$ **the adjacency matrix**, ϱ_{comb} the **combinatorial distance**. Then:

$$\sum b(u,v)\varrho_{\mathrm{comb}}(u,v)^2 = \sum_{u\sim v} 1 = \deg(v).$$

- ϱ_{comb} is intrinsic for $m = \deg$, i.e., for L_{norm}.
- **Not intrinsic for L_{comb}!** However, **equivalent to intrinsic** $\Leftrightarrow$ $\sup\deg < \infty$!

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Intrinsic metrics **recover many results from manifolds** for graphs!

M. Keller, D. Lenz, & R. Wojciechowski, *Graphs and Discrete Dirichlet Spaces*, in print, Springer, 2021.

But: Each $(\mathcal{V}, m; b)$ has infinitely many intrinsic metrics! No "maximal" metric..

Another problem: **how to construct an intrinsic metric?**

Intrinsic metric for Kirchhoff Laplacians

Quadratic form (Energy form/Dirichlet integral)

$$\mathfrak{Q}[f] := \int_{\mathcal{G}} |\nabla f|^2 \nu(dx) \quad \Big(= \langle \Delta f, f \rangle_{L^2(\mu)} \quad \text{for } f \in \mathrm{dom}(\Delta_{\mathrm{Kir},0}) \Big)$$

It is a **strongly local** Dirichlet form in $L^2(\mathcal{G}; \mu)$.

- Background: To each strongly local Dirichlet form, one can associate its **intrinsic metric** $\Rightarrow$ generalize results from Riemannian manifolds!

Definition (intrinsic metric for $(\mathcal{G}, \mu, \nu)$)

$$\varrho_{\mathrm{intr}}(x, y) = \sup \big\{ f(x) - f(y) \,|\, f \in \mathcal{D}_{\mathrm{loc}} \big\}, \qquad x, y \in \mathcal{G},$$

$$\mathcal{D}_{\mathrm{loc}} = \big\{ f \in H^1_{\mathrm{loc}}(\mathcal{G}) \,\big|\, \nu(x) |\nabla f(x)|^2 \le \mu(x) \ \text{ for a.e. } x \in \mathcal{G} \big\}.$$

K.-T. Sturm, *Analysis on local Dirichlet spaces I – III*, (1994–1996).

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It is a **strongly local** Dirichlet form in $L^2(\mathcal{G};\mu)$.

Since both μ, ν are **edgewise constant**,

$$\varrho_{\mathrm{intr}}(x,y) = \varrho_\eta(x,y) := \inf_{\mathcal{P}} \int_{\mathcal{P}} \eta(dx) = \inf_{\mathcal{P}} \int_{\mathcal{P}} \sqrt{\frac{\mu}{\nu}} dx.$$

If $(\mathcal{V}, m; b)$ is the graph associated with $(\mathcal{G}, \mu, \nu)$, then the **induced metric**

$$\varrho_{\mathcal{V}}(u,v) := \varrho_\eta(u,v), \qquad u, v \in \mathcal{V}$$

is intrinsic w.r.t. $(\mathcal{V}, m; b)$!

Manifolds → local Dirichlet forms → discrete measure spaces

From discrete graphs to metric graphs?

A **cable system** for $(\mathcal{V}, m; b)$ is a weighted metric graph $(\mathcal{G}, \mu, \nu)$ s.t.

$$L_{\mathcal{V},m,b} = L(\mathcal{G}, \mu, \nu),$$

i.e., the previous construction gives the discrete Laplacian $L_{\mathcal{V},m,b}$.

Theorem

(i) Every locally finite $(\mathcal{V}, m; b)$ **has a cable system**.

(ii) For every $(\mathcal{V}, m; b)$ equipped with a **finite jump size** intrinsic metric ϱ there is **finite intrinsic size** cable system such that $\varrho = \varrho_{\mathcal{V}} = \varrho_\eta|_{\mathcal{V}\times\mathcal{V}}$.
(Finite jump size = no arbitrarily long edge w.r.t. ϱ)

WARNING: Upon some normalization (e.g., **canonical CS**), (almost!) a **bijection** between **cable systems** and **intrinsic path metrics** for $(\mathcal{V}, m; b)$

To construct an intrinsic metric $\cong$ To construct a cable system

M.Folz, *Volume growth and stochastic completeness of graphs*, TAMS(2014)

Quasi-isometries

Definition

A map $\phi\colon X_1 \to X_2$ between two metric spaces (X_1, ϱ_1) and (X_2, ϱ_2) is called a **quasi-isometry** if there are $a, b, R > 0$ s.t.

$$a^{-1}(\varrho_1(x,y) - b) \le \varrho_2(\phi(x), \phi(y)) \le a(\varrho_1(x,y) + b),$$

for all $x, y \in X_1$ and, moreover, $\bigcup_{x\in X_1} B_R(\phi(x); \varrho_2) = X_2$.

Examples (The Švarc–Milnor Lemma)

- Cayley graph of $\pi_1(M)$ and the universal cover $\widetilde{M}$ of a compact manifold M,
- Cayley graph and the corresponding equilateral metric graph.

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Corollary

Let $(\mathcal{G}, \mu, \nu)$ be a cable system for $(\mathcal{V}, m; b)$. The metric spaces $(\mathcal{G}, \varrho_\eta)$ and $(\mathcal{V}, \varrho_\mathcal{V})$ are **quasi-isometric** if and only if $(\mathcal{G}, \mu, \nu)$ has **finite intrinsic size**.

For $(\mathcal{V}, m; b)$ with an intrinsic metric ϱ, a **cable system** is a **quasi-isometric length space with the same combinatorial structure**

$\Rightarrow$ **connections between their large scale/global properties**!

Applications: Self-adjointness (a.k.a. Quantum Completeness)

On $(\mathcal{G}, \mu, \nu)$ we introduced Laplacians Δ_{Kir} and $\Delta_{\mathrm{Kir},0} = \overline{\Delta_{\mathrm{Kir}} \restriction C_c}^{\|\cdot\|_{L^2(\mathcal{G};\mu)}}$.

Δ_{Kir} is self-adjoint $\Leftrightarrow$ $\Delta_{\mathrm{Kir},0} = \Delta_{\mathrm{Kir}}$ ($\Leftrightarrow$ L^2**-uniqueness for Schrödinger/Wave eq.**)

Problem: Do we need a boundary condition at "infinity"?

When $\Delta_{\mathrm{Kir},0} = \Delta_{\mathrm{Kir}}$?

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von Neumann formulas

$$\mathrm{dom}(\Delta_{\mathrm{Kir}}) = \mathrm{dom}(\Delta_D) \dot{+} \ker(\Delta_{\mathrm{Kir}} + \lambda), \quad \lambda \in \mathbb{C} \setminus \sigma(-\Delta_D).$$

Δ_D is the Dirichlet Laplacian (the Friedrichs extension of $\Delta_{\mathrm{Kir},0}$)

Since $\ker(\Delta_{\mathrm{Kir}} - \lambda) = L^2$ λ-harmonic functions and $\sigma(-\Delta_D) \subseteq [0, \infty)$:

- self-adjoint uniqueness $\Leftrightarrow$ no L^2 harmonic f-ns (λ-harmonic with $\lambda > 0$),
- description of self-adjoint extns = description of L^2 λ-harmonic functions!

Graph Boundaries

Poisson = bounded harmonic; Martin = positive harmonic, ...

Applications: Self-adjointness (a.k.a. Quantum Completeness)

Gaffney-type Theorem on Metric graphs

If $(\mathcal{G}, \varrho_\eta)$ is complete, then $\Delta_{\mathrm{Kir},0} = \Delta_{\mathrm{Kir}}$.

For manifolds: Cauchy boundary $\partial_C M = \overline{M} \setminus M$; completeness is $\partial_C M = \emptyset$
completeness $\Rightarrow$ self-adjoint uniqueness (Gaffney'54; Roelcke'60; Chernoff'73).

Proof: Assume the converse: $\exists\ u \in L^2(\mathcal{G}; \mu)$ such that $u \neq 0$ is λ-harmonic, $\lambda > 0$. However, $|u| \geq 0$ is subharmonic.
By a version of Yau's L^p-Liouville theorem for strongly local Dirichlet forms, $|u| \equiv 0$ if $(\mathcal{G}, \varrho_\eta)$ is complete. Contradiction. □

K.-T. Sturm, *Analysis on local Dirichlet spaces I*, Crelle's J. (1994).

• **Stability under semi-bounded perturbations**
("completeness w.r.t. intrinsic metric+semiboundedness $\Rightarrow$ quantum compl.")

WARNING: Self-adjointness is open for $(\mathcal{G}_d, |\cdot|, \mu, \nu)$ even if $\mathcal{G}_d = \mathbb{Z}^2$...

Applications: Self-adjointness (a.k.a. Quantum Completeness)

Gaffney-type Theorem on Metric graphs

If $(\mathcal{G}, \varrho_\eta)$ is complete, then $\Delta_{\mathrm{Kir},0} = \Delta_{\mathrm{Kir}}$.

For manifolds: Cauchy boundary $\partial_C M = \overline{M} \setminus M$; completeness is $\partial_C M = \emptyset$
completeness $\Rightarrow$ self-adjoint uniqueness (Gaffney'54; Roelcke'60; Chernoff'73).

Corollary (Gaffney-type Theorem on graphs)

If ϱ is a path metric, intrinsic w.r.t. $(\mathcal{V}, m; b)$, and $(\mathcal{V}, \varrho)$ is complete, then $L_{\mathcal{V},m,b}$ is self-adjoint in $\ell^2(\mathcal{V}; m)$.

Proof: $\exists$ cable system $(\mathcal{G}, \mu, \nu)$ s.t. $\varrho = \varrho_\eta$ on $\mathcal{V}$; $(\mathcal{G}, \varrho_\eta)$ is complete if $(\mathcal{V}, \varrho)$ is complete (e.g., by the Hopf–Rinow Theorem for length spaces, then by quasi-isometry to weighted graphs from metric graphs). □

X. Huang, M. Keller, J. Masamune and R. Wojciechowski, *A note on self-adjoint extensions of the Laplacian on weighted graphs*, J. Funct. Anal. (2013)

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WARNING: The above result is **not enough** to show that L_{comb} is self-adjoint!

Applications: Recurrence

G. Pólya (1921): **A (simple) random walk on $\mathbb{Z}^d$ is recurrent $\Leftrightarrow$ $d \leq 2$**

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Recurrence can be defined via:

- behavior of a heat kernel for large times, (in $\mathbb{R}^d$, heat kernel $\approx t^{-d/2}$)
- behavior of the Green's function at zero energy

— in **Quantum Mechanics** = *zero energy resonance/weak bound state/virtual pole*

- every nonnegative superharmonic function is constant

- Recurrence on Riemann surfaces: for simply connected, **the type problem**.
- $\pi_1(M)$ is recurrent $\Leftrightarrow$ the universal cover $\widetilde{M}$ of M is recurrent (Varopoulos'83)
- M is recurrent if **"not enough volume"** (Grigor'yan, Karp, Varopoulos'82-83)
  — extension to strongly local Dirichlet forms by STURM (1994)

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Remark: As with the self-adjointness, from metric to weighted graphs, e.g., discrete recurrence volume test, Karp-type theorem etc.

B. Hua, M. Keller, *Harmonic functions of general graph Laplacians*, Calc.Var.(2014)

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- M. Gromov (Groups of polynomial growth), and
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Theorem (Varopoulos, 1985)

G is **recurrent** $\Leftrightarrow$ G contains a finite index subgroup **isomorphic** either to $\mathbb{Z}$ or $\mathbb{Z}^2$

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Theorem (AK–NICOLUSSI'21)

$\mathcal{G}_C = \mathcal{C}(\mathsf{G}, S)$ a Cayley graph and $(\mathcal{G}_C, \mu, \nu)$ a weighted metric graph. Then $(\mathcal{G}_C, \mu, \nu)$ is **recurrent** $\Leftrightarrow$ the **discrete-time random walk** on $\mathcal{G}_C$ generated by $b\colon \mathcal{E}_C \to \mathbb{R}_{>0}$ with $b(e) := \frac{\nu(e)}{|e|}$ is **recurrent**.

In particular, if G is recurrent and $\sup_{e\in\mathcal{E}} \frac{\nu(e)}{|e|} < \infty$, then $(\mathcal{G}_C, \mu, \nu)$ is recurrent.

Applications: Ultracontractivity and CLR estimates

Let Δ_D be the **Dirichlet Laplacian** on $(\mathcal{G}_\mathcal{C}, \mu, \nu)$ with $\mu = \nu \equiv 1$.

Theorem (AK–Nicolussi'21)

(i) If G is not recurrent and $\gamma_\mathsf{G}(n) \asymp n^N$, then $\|\mathrm{e}^{t\Delta_D}\|_{1\to\infty} \lesssim t^{-N/2}$ $\forall t > 0$ whenever $\sup|e| < \infty$. Here γ_G is the growth function.

(ii) If G is not virtually nilpotent and $\sup|e| < \infty$, the above est. holds $\forall N > 0$.

(iii) If G is recurrent and $\inf|e| > 0$, then $\|\mathrm{e}^{t\Delta_D}\|_{1\to\infty} \gtrsim t^{-1}$.

Corollary (AK–Nicolussi'21): Let $\mathsf{H}_V := -\Delta_D - V(x)$, $V\colon \mathcal{G} \to \mathbb{R}$

(i) If G is recurrent, $\inf|e| > 0$, and $0 \le V \in C_c(\mathcal{G})$, then H_V admits at least one negative e.v.

(ii) If G is not recurrent, $\gamma_\mathsf{G}(n) \asymp n^N$, and $\sup|e| < \infty$, then for $V \ge 0$

$$\dim\left(\operatorname{ran} 1_{(-\infty,0)}(\mathsf{H}_V)\right) \lesssim_\mathcal{G} \int_\mathcal{G} V(x)^{N/2} dx.$$

Thank you for your attention!