On coverings and perfect colorings of hypergraphs

Anna Taranenko

Sobolev Institute of Mathematics, Novosibirsk, Russia

8th European Congress of Mathematics
Portorož, Slovenia, 2021

Perfect coloring of graphs

A perfect k-coloring (equitable k-partition) is a function f from the vertex set to colors $\{1, \ldots, k\}$ such that each vertex of color i is adjacent to exactly $s_{i,j}$ vertices of color j.

Perfect coloring of graphs

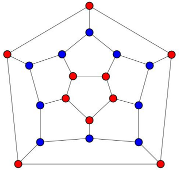
A perfect k-coloring (equitable k-partition) is a function f from the vertex set to colors $\{1, \ldots, k\}$ such that each vertex of color i is adjacent to exactly $s_{i,j}$ vertices of color j.

$S = (s_{i,j})$ is the parameter matrix of a perfect coloring.

Perfect coloring of graphs

A perfect k-coloring (equitable k-partition) is a function f from the vertex set to colors $\{1, \dots, k\}$ such that each vertex of color i is adjacent to exactly $s_{i,j}$ vertices of color j.

$S = (s_{i,j})$ is the parameter matrix of a perfect coloring.



$$S = \begin{array}{c|cc} & r & b \\ \hline r & 2 & 1 \\ b & 1 & 2 \end{array}$$

Eigenvalues and coverings of graphs

Adjacency matrix M of a graph G: $m_{i,j} = 1$ if (i,j) is an edge, $m_{i,j} = 0$ otherwise.

Eigenvalues and coverings of graphs

Adjacency matrix M of a graph G: $m_{i,j} = 1$ if (i, j) is an edge, $m_{i,j} = 0$ otherwise.

Perfect coloring is a $(0, 1)$-matrix P, $p_{v,j} = 1 \Leftrightarrow f(v) = j$, such that

$$MP = PS.$$

Eigenvalues and eigenvectors of a graph G are eigenvalues and eigenvectors of the adjacency matrix M.

Eigenvalues and coverings of graphs

Adjacency matrix M of a graph G: $m_{i,j} = 1$ if (i, j) is an edge, $m_{i,j} = 0$ otherwise.

Perfect coloring is a $(0, 1)$-matrix P, $p_{v,j} = 1 \Leftrightarrow f(v) = j$, such that

$$MP = PS.$$

Eigenvalues and eigenvectors of a graph G are eigenvalues and eigenvectors of the adjacency matrix M.

Every eigenvalue of the parameter matrix S of a perfect coloring is an eigenvalue of the adjacency matrix M.

Eigenvalues and coverings of graphs

Adjacency matrix M of a graph G: $m_{i,j} = 1$ if (i, j) is an edge, $m_{i,j} = 0$ otherwise.

Perfect coloring is a $(0, 1)$-matrix P, $p_{v,j} = 1 \Leftrightarrow f(v) = j$, such that

$$MP = PS.$$

Eigenvalues and eigenvectors of a graph G are eigenvalues and eigenvectors of the adjacency matrix M.

Every eigenvalue of the parameter matrix S of a perfect coloring is an eigenvalue of the adjacency matrix M.

A covering of a graph H by a graph G = a perfect coloring of G with the parameter matrix equal to the adjacency matrix of H.

Eigenvalues and coverings of graphs

Adjacency matrix M of a graph G: $m_{i,j} = 1$ if (i, j) is an edge, $m_{i,j} = 0$ otherwise.

Perfect coloring is a $(0, 1)$-matrix P, $p_{v,j} = 1 \Leftrightarrow f(v) = j$, such that

$$MP = PS.$$

Eigenvalues and eigenvectors of a graph G are eigenvalues and eigenvectors of the adjacency matrix M.

Every eigenvalue of the parameter matrix S of a perfect coloring is an eigenvalue of the adjacency matrix M.

A covering of a graph H by a graph G = a perfect coloring of G with the parameter matrix equal to the adjacency matrix of H.

Our goal: develop the similar concepts for hypergraphs

Hypergraphs, incidence matrices, bipartite representation

$\mathcal{G}(X, W)$ is a hypergraph,
X is the vertex set, $|X| = n$, W is the hyperedge set, $|W| = m$.

Hypergraphs, incidence matrices, bipartite representation

$\mathcal{G}(X, W)$ is a hypergraph,
X is the vertex set, $|X| = n$, W is the hyperedge set, $|W| = m$.

The incidence matrix $\mathbb{I}$ of $\mathcal{G}$ is an $(n \times m)$-rectangular $(0, 1)$-matrix with a (x, w)-entry equals $1 \Leftrightarrow x \in w$ in $\mathcal{G}$.

Hypergraphs, incidence matrices, bipartite representation

$\mathcal{G}(X, W)$ is a hypergraph,
X is the vertex set, $|X| = n$, W is the hyperedge set, $|W| = m$.

The incidence matrix $\mathbb{I}$ of $\mathcal{G}$ is an $(n \times m)$-rectangular $(0, 1)$-matrix with a (x, w)-entry equals $1 \Leftrightarrow x \in w$ in $\mathcal{G}$.

A degree $deg(S)$ of $S \subset X$ is the number of hyperedges, containing all vertices from S.
A hypergraph is d-uniform if each hyperedge consists of exactly d vertices.

Hypergraphs, incidence matrices, bipartite representation

$\mathcal{G}(X, W)$ is a hypergraph,
X is the vertex set, $|X| = n$, W is the hyperedge set, $|W| = m$.

The incidence matrix $\mathbb{I}$ of $\mathcal{G}$ is an $(n \times m)$-rectangular $(0, 1)$-matrix with a (x, w)-entry equals $1 \Leftrightarrow x \in w$ in $\mathcal{G}$.

A degree $deg(S)$ of $S \subset X$ is the number of hyperedges, containing all vertices from S.
A hypergraph is d-uniform if each hyperedge consists of exactly d vertices.

The bipartite representation $G(X, W; E)$ of a hypergraph $\mathcal{G}(X, W)$ is a bipartite graph, x is adjacent to w in G iff x is incident to w in $\mathcal{G}$.
The adjacency matrix M_G of the bipartite representation is

$$M_G = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I}^T & 0 \end{pmatrix}.$$

Plain adjacency matrix of a hypergraph

Let $\mathcal{G}$ be a hypergraph with the incidence matrix $\mathbb{I}$.

Simple graph is a 2-uniform hypergraph. The adjacency matrix of a graph is $M = \mathbb{I}\mathbb{I}^T - D$, where D is the diagonal degree matrix.

Plain adjacency matrix of a hypergraph

Let $\mathcal{G}$ be a hypergraph with the incidence matrix $\mathbb{I}$.

Simple graph is a 2-uniform hypergraph. The adjacency matrix of a graph is $M = \mathbb{I}\mathbb{I}^T - D$, where D is the diagonal degree matrix.

Let the plain adjacency matrix of a hypergraph $\mathcal{G}$ be the matrix $M = \mathbb{I}\mathbb{I}^T - D$.

Plain eigenvalues and eigenvectors of a hypergraph are eigenvalues and eigenvectors of its plain adjacency matrix.

Plain adjacency matrix of a hypergraph

Let $\mathcal{G}$ be a hypergraph with the incidence matrix $\mathbb{I}$.

Simple graph is a 2-uniform hypergraph. The adjacency matrix of a graph is $M = \mathbb{I}\mathbb{I}^T - D$, where D is the diagonal degree matrix.

Let the plain adjacency matrix of a hypergraph $\mathcal{G}$ be the matrix $M = \mathbb{I}\mathbb{I}^T - D$.
Plain eigenvalues and eigenvectors of a hypergraph are eigenvalues and eigenvectors of its plain adjacency matrix.

Such approach to the adjacency matrix was used for studying Berge cycles, metric and expanding properties of hypergraphs.

Plain adjacency matrix of a hypergraph

Let $\mathcal{G}$ be a hypergraph with the incidence matrix $\mathbb{I}$.

Simple graph is a 2-uniform hypergraph. The adjacency matrix of a graph is $M = \mathbb{I}\mathbb{I}^T - D$, where D is the diagonal degree matrix.

Let the plain adjacency matrix of a hypergraph $\mathcal{G}$ be the matrix $M = \mathbb{I}\mathbb{I}^T - D$.
Plain eigenvalues and eigenvectors of a hypergraph are eigenvalues and eigenvectors of its plain adjacency matrix.

Such approach to the adjacency matrix was used for studying Berge cycles, metric and expanding properties of hypergraphs.

Another approach: the adjacency matrix of a d-uniform hypergraph is a d-dimensional matrix.

Multidimensional adjacency matrices

A *d-dimensional matrix A of order n* is an array (a_α), $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{1, \ldots, n\}$.

Multidimensional adjacency matrices

A d-dimensional matrix A of order n is an array (a_α), $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{1, \dots, n\}$.

1. Combinatorial approach: The adjacency matrix $\mathbb{M}$ of a d-uniform hypergraph $\mathcal{G}$ on n vertices is a d-dimensional $(0,1)$-matrix of order n with entries $m_\alpha = 1 \Leftrightarrow (\alpha_1, \dots, \alpha_d)$ is a hyperedge in $\mathcal{G}$.

Multidimensional adjacency matrices

A d-dimensional matrix A of order n is an array (a_α), $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{1, \dots, n\}$.

1. Combinatorial approach: The adjacency matrix $\mathbb{M}$ of a d-uniform hypergraph $\mathcal{G}$ on n vertices is a d-dimensional $(0,1)$-matrix of order n with entries $m_\alpha = 1 \Leftrightarrow (\alpha_1, \dots, \alpha_d)$ is a hyperedge in $\mathcal{G}$.

2. Algebraic approach: For 2-dimensional matrices $B^1, \dots, B^d$ of order n define folding $C = [B^1, \dots, B^d]$ to be the d-dimensional matrix of order n:

$$c_{\alpha_1,\dots,\alpha_d} = \sum_{i=1}^{n} b^1_{\alpha_1,i} \cdots b^d_{\alpha_d,i}.$$

Multidimensional adjacency matrices

A *d-dimensional matrix A of order n* is an array (a_α), $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{1, \dots, n\}$.

1. Combinatorial approach: The adjacency matrix $\mathbb{M}$ of a d-uniform hypergraph $\mathcal{G}$ on n vertices is a d-dimensional $(0,1)$-matrix of order n with entries $m_\alpha = 1 \Leftrightarrow (\alpha_1, \dots, \alpha_d)$ is a hyperedge in $\mathcal{G}$.

2. Algebraic approach: For 2-dimensional matrices $B^1, \dots, B^d$ of order n define folding $C = [B^1, \dots, B^d]$ to be the d-dimensional matrix of order n:

$$c_{\alpha_1,\dots,\alpha_d} = \sum_{i=1}^{n} b^1_{\alpha_1,i} \cdots b^d_{\alpha_d,i}.$$

The full adjacency matrix $\mathbb{A}$ of a d-uniform hypergraph $\mathcal{G}$ on n vertices is

$$\mathbb{A} = [\mathbb{I}, \dots, \mathbb{I}].$$

Full adjacency matrices and totally regular hypergraphs

Proposition

Let $\mathbb{A}$ be the full adjacency matrix of a d-uniform hypergraph. Then entries a_α are exactly the degrees of sets $S(\alpha) = \{\alpha_1, \dots, \alpha_d\}$.

Full adjacency matrices and totally regular hypergraphs

Proposition

Let $\mathbb{A}$ be the full adjacency matrix of a d-uniform hypergraph. Then entries a_α are exactly the degrees of sets $S(\alpha) = \{\alpha_1, \dots, \alpha_d\}$.

We will say that a d-uniform hypergraph $\mathcal{G} = (X, W)$ is totally $(r_1, \dots, r_{d-1})$-regular if every $S \subset X$, $|S| = i$, has $deg(S) = r_i$.

Full adjacency matrices and totally regular hypergraphs

Proposition

Let $\mathbb{A}$ be the full adjacency matrix of a d-uniform hypergraph. Then entries a_α are exactly the degrees of sets $S(\alpha) = \{\alpha_1, \dots, \alpha_d\}$.

We will say that a d-uniform hypergraph $\mathcal{G} = (X, W)$ is totally $(r_1, \dots, r_{d-1})$-regular if every $S \subset X$, $|S| = i$, has $deg(S) = r_i$.

Proposition

If $\mathcal{G}$ is a d-uniform totally $(r_1, \dots, r_{d-1})$-regular hypergraph, then

$$\mathbb{A} = \mathbb{M} + \sum_{t=1}^{d-1} r_t \mathcal{I}_t,$$

where $\mathcal{I}_t$ is a d-dimensional $(0,1)$-matrix, whose unity entries indexed by $(\alpha_1, \dots, \alpha_d)$ with exactly t different components.

Products of multidimensional matrices

The following multidimensional matrix products and eigenvalues were studied by L. Qi, L.H. Lim, S. Hu, C. Ling, J.-Yu Shao, Z. Huang, J. Cooper, A. Dutle, ...

Products of multidimensional matrices

The following multidimensional matrix products and eigenvalues were studied by L. Qi, L.H. Lim, S. Hu, C. Ling, J.-Yu Shao, Z. Huang, J. Cooper, A. Dutle, ...

Let A be d-dimensional matrix of order n. Define product $\circ$:

- If v is a vector, then $A \circ v$ is a vector u:

$$u_j = \sum_{i_1,\dots,i_{d-1}=1}^{n} a_{j,i_1,\dots,i_{d-1}} v_{i_1} \cdots v_{i_{d-1}}.$$

Products of multidimensional matrices

The following multidimensional matrix products and eigenvalues were studied by L. Qi, L.H. Lim, S. Hu, C. Ling, J.-Yu Shao, Z. Huang, J. Cooper, A. Dutle, ...

Let A be d-dimensional matrix of order n. Define product $\circ$:

- If v is a vector, then $A \circ v$ is a vector u:

$$u_j = \sum_{i_1,\dots,i_{d-1}=1}^{n} a_{j,i_1,\dots,i_{d-1}} v_{i_1} \cdots v_{i_{d-1}}.$$

- If P is a 2-dimensional matrix, then $A \circ P$ is a d-dimensional matrix C:

$$C_{j,k_1\dots,k_{d-1}} = \sum_{i_1,\dots,i_{d-1}=1}^{n} a_{j,i_1,\dots,i_{d-1}} p_{k_1,i_1} \cdots p_{k_{d-1},i_{d-1}}.$$

Products of multidimensional matrices

The following multidimensional matrix products and eigenvalues were studied by L. Qi, L.H. Lim, S. Hu, C. Ling, J.-Yu Shao, Z. Huang, J. Cooper, A. Dutle, ...

Let A be d-dimensional matrix of order n. Define product $\circ$:

- If v is a vector, then $A \circ v$ is a vector u:

$$u_j = \sum_{i_1,\ldots,i_{d-1}=1}^{n} a_{j,i_1,\ldots,i_{d-1}} v_{i_1} \cdots v_{i_{d-1}}.$$

- If P is a 2-dimensional matrix, then $A \circ P$ is a d-dimensional matrix C:

$$C_{j,k_1\ldots,k_{d-1}} = \sum_{i_1,\ldots,i_{d-1}=1}^{n} a_{j,i_1,\ldots,i_{d-1}} p_{k_1,i_1} \cdots p_{k_{d-1},i_{d-1}}.$$

- If B a t-dimensional matrix of order n, then $A \circ B$ is a similar $((d-1)(t-1)+1)$-dimensional matrix.

Eigenvalues of hypergraphs

λ is an eigenvalue and v is the eigenvector of a d-dimensional matrix A if

$$A \circ v = \lambda(\mathcal{I} \circ v),$$

where $\mathcal{I} = \mathcal{I}_1$ is the d-dimensional identity matrix.

Eigenvalues of hypergraphs

λ is an eigenvalue and v is the eigenvector of a d-dimensional matrix A if

$$A \circ v = \lambda(\mathcal{I} \circ v),$$

where $\mathcal{I} = \mathcal{I}_1$ is the d-dimensional identity matrix.

Let $\mathcal{G}$ be a d-uniform hypergraph.
Eigenvalues and eigenvectors of $\mathcal{G}$ are eigenvalues and eigenvectors of the adjacency matrix $\mathbb{M}$.

Eigenvalues of hypergraphs

λ is an eigenvalue and v is the eigenvector of a d-dimensional matrix A if

$$A \circ v = \lambda(\mathcal{I} \circ v),$$

where $\mathcal{I} = \mathcal{I}_1$ is the d-dimensional identity matrix.

Let $\mathcal{G}$ be a d-uniform hypergraph.
Eigenvalues and eigenvectors of $\mathcal{G}$ are eigenvalues and eigenvectors of the adjacency matrix $\mathbb{M}$.
Full eigenvalues and eigenvectors of $\mathcal{G}$ are eigenvalues and eigenvectors of the full adjacency matrix $\mathbb{A}$.

Eigenvalues of hypergraphs

λ is an eigenvalue and v is the eigenvector of a d-dimensional matrix A if

$$A \circ v = \lambda(\mathcal{I} \circ v),$$

where $\mathcal{I} = \mathcal{I}_1$ is the d-dimensional identity matrix.

Let $\mathcal{G}$ be a d-uniform hypergraph.
Eigenvalues and eigenvectors of $\mathcal{G}$ are eigenvalues and eigenvectors of the adjacency matrix $\mathbb{M}$.
Full eigenvalues and eigenvectors of $\mathcal{G}$ are eigenvalues and eigenvectors of the full adjacency matrix $\mathbb{A}$.

Theorem (T., 2021+)

Let v be a plain eigenvector for $\mathcal{G}$. Then

- v is a full eigenvector for $\mathcal{G}$;
- If $\mathcal{G}$ is a totally regular hypergraph, then v is an eigenvector for $\mathcal{G}$.

In all cases, eigenvalues of one type can be counted from another.

Perfect colorings of hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform hypergraph with the incidence matrix $\mathbb{I}$.

Perfect colorings of hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform hypergraph with the incidence matrix $\mathbb{I}$.
Direct definition: A function $f : X \to \{1, \ldots, k\}$ is a perfect k-coloring of $\mathcal{G}$ if a coloring of a vertex uniquely defines the coloring of all incident hyperedges.

Perfect colorings of hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform hypergraph with the incidence matrix $\mathbb{I}$.

Direct definition: A function $f : X \to \{1, \ldots, k\}$ is a perfect k-coloring of $\mathcal{G}$ if a coloring of a vertex uniquely defines the coloring of all incident hyperedges.

Using bipartite representation: A perfect k-coloring of a $\mathcal{G}$ is given by vertex k-coloring matrix P and hyperedge coloring matrix R satisfying

$$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I}^T & 0 \end{pmatrix} \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} = \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}.$$

S is the HV-parameter matrix; T is the VH-parameter matrix.

Perfect colorings of hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform hypergraph with the incidence matrix $\mathbb{I}$.

Direct definition: A function $f : X \to \{1, \ldots, k\}$ is a perfect k-coloring of $\mathcal{G}$ if a coloring of a vertex uniquely defines the coloring of all incident hyperedges.

Using bipartite representation: A perfect k-coloring of a $\mathcal{G}$ is given by vertex k-coloring matrix P and hyperedge coloring matrix R satisfying

$$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I}^T & 0 \end{pmatrix} \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} = \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}.$$

S is the HV-parameter matrix; T is the VH-parameter matrix.

Multidimensional definition: A vertex k-coloring matrix P defines a perfect coloring of $\mathcal{G}$ if

$$\mathbb{A} \circ P = P \circ \mathbb{S}.$$

Parameter matrix $\mathbb{S}$ is a d-dimensional matrix of order k.

Perfect colorings of hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform hypergraph with the incidence matrix $\mathbb{I}$.

Direct definition: A function $f : X \to \{1, \ldots, k\}$ is a perfect k-coloring of $\mathcal{G}$ if a coloring of a vertex uniquely defines the coloring of all incident hyperedges.

Using bipartite representation: A perfect k-coloring of a $\mathcal{G}$ is given by vertex k-coloring matrix P and hyperedge coloring matrix R satisfying

$$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I}^T & 0 \end{pmatrix} \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} = \begin{pmatrix} 0 & P \\ R & 0 \end{pmatrix} \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}.$$

S is the HV-parameter matrix; T is the VH-parameter matrix.

Multidimensional definition: A vertex k-coloring matrix P defines a perfect coloring of $\mathcal{G}$ if

$$\mathbb{A} \circ P = P \circ \mathbb{S}.$$

Parameter matrix $\mathbb{S}$ is a d-dimensional matrix of order k.

All three definitions are equivalent.

Transversals in hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform r-regular hypergraph.
A t-transversal in $\mathcal{G}$ is a set $U \subseteq X$ such that each hyperedge w contains exactly t vertices from U.

Transversals in hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform r-regular hypergraph.
A t-transversal in $\mathcal{G}$ is a set $U \subseteq X$ such that each hyperedge w contains exactly t vertices from U.

If U is a t-transversal in $\mathcal{G}$, then U and $X \setminus U$ are the color classes of a perfect 2-coloring of $\mathcal{G}$.

Transversals in hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform r-regular hypergraph.
A t-transversal in $\mathcal{G}$ is a set $U \subseteq X$ such that each hyperedge w contains exactly t vertices from U.

If U is a t-transversal in $\mathcal{G}$, then U and $X \setminus U$ are the color classes of a perfect 2-coloring of $\mathcal{G}$.

HV- and VH-parameter matrices:

$$S = \begin{pmatrix} t & d - t \end{pmatrix}; \quad T = \begin{pmatrix} r \\ r \end{pmatrix}.$$

Transversals in hypergraphs

Let $\mathcal{G} = (X, W)$ be a d-uniform r-regular hypergraph.
A t-transversal in $\mathcal{G}$ is a set $U \subseteq X$ such that each hyperedge w contains exactly t vertices from U.

If U is a t-transversal in $\mathcal{G}$, then U and $X \setminus U$ are the color classes of a perfect 2-coloring of $\mathcal{G}$.

HV- and VH-parameter matrices:

$$S = \begin{pmatrix} t & d-t \end{pmatrix}; \quad T = \begin{pmatrix} r \\ r \end{pmatrix}.$$

The parameter matrix $\mathbb{S}$:

$$S_\alpha = r \prod_{i=2}^{d} t^{\alpha_i} (d-t)^{1-\alpha_i}; \;\; \alpha_i \in \{0, 1\}.$$

Parameter matrix of a hypergraph

Let $\mathcal{G}$ be a d-uniform hypergraph with the full adjacency matrix $\mathbb{A}$.
P is a perfect k-coloring of $\mathcal{G}$ if

$$\mathbb{A} \circ P = P \circ \mathbb{S}.$$

$\mathbb{S}$ is the parameter matrix of the perfect coloring.

Parameter matrix of a hypergraph

Let $\mathcal{G}$ be a d-uniform hypergraph with the full adjacency matrix $\mathbb{A}$.
P is a perfect k-coloring of $\mathcal{G}$ if

$$\mathbb{A} \circ P = P \circ \mathbb{S}.$$

$\mathbb{S}$ is the parameter matrix of the perfect coloring.

Theorem (T., 2021+)

S is the HV-parameter matrix and T is the VH-parameter matrix of a perfect coloring of a hypergraph $\mathcal{G}$ if and only if

$$\mathbb{S} = [T, S^T, \ldots, S^T].$$

Properties of parameter matrices

In case of totally regular hypergraphs, perfect colorings can be defined as through the adjacency matrix, as the full adjacency matrix.

Properties of parameter matrices

In case of totally regular hypergraphs, perfect colorings can be defined as through the adjacency matrix, as the full adjacency matrix.

Proposition

If $\mathcal{G}$ is a uniform totally regular hypergraph, P is a vertex coloring matrix, then

$$\mathbb{A} \circ P = P \circ \mathbb{S} \Leftrightarrow \mathbb{M} \circ P = P \circ \mathbb{T}.$$

Properties of parameter matrices

In case of totally regular hypergraphs, perfect colorings can be defined as through the adjacency matrix, as the full adjacency matrix.

Proposition

If $\mathcal{G}$ is a uniform totally regular hypergraph, P is a vertex coloring matrix, then

$$\mathbb{A} \circ P = P \circ \mathbb{S} \Leftrightarrow \mathbb{M} \circ P = P \circ \mathbb{T}.$$

Theorem (T., 2021+)

Let P be a perfect coloring of a hypergraph $\mathcal{G}$ with the parameter matrix $\mathbb{S}$. If λ and v are eigenvalue and eigenvector of $\mathbb{S}$, then λ and v are eigenvalue and eigenvector of the full adjacency matrix $\mathbb{A}$.

Properties of parameter matrices

In case of totally regular hypergraphs, perfect colorings can be defined as through the adjacency matrix, as the full adjacency matrix.

Proposition

If $\mathcal{G}$ is a uniform totally regular hypergraph, P is a vertex coloring matrix, then

$$\mathbb{A} \circ P = P \circ \mathbb{S} \Leftrightarrow \mathbb{M} \circ P = P \circ \mathbb{T}.$$

Theorem (T., 2021+)

Let P be a perfect coloring of a hypergraph $\mathcal{G}$ with the parameter matrix $\mathbb{S}$. If λ and v are eigenvalue and eigenvector of $\mathbb{S}$, then λ and v are eigenvalue and eigenvector of the full adjacency matrix $\mathbb{A}$.

Theorem (T., 2021+)

The parameter matrix $\mathbb{S}$ of a perfect coloring of a hypergraph can be symmetrized by the vector of color densities.

Minimal coloring

Theorem (Weisfeiler, Leman, 1968)

Let G be a graph. Then there is the minimal perfect coloring f such that every other perfect coloring of G is obtained from f by splitting of color classes.

Minimal coloring

Theorem (Weisfeiler, Leman, 1968)

Let G be a graph. Then there is the minimal perfect coloring f such that every other perfect coloring of G is obtained from f by splitting of color classes.

For regular graphs the minimal perfect coloring is monochromatic.

Minimal coloring

Theorem (Weisfeiler, Leman, 1968)

Let G be a graph. Then there is the minimal perfect coloring f such that every other perfect coloring of G is obtained from f by splitting of color classes.

For regular graphs the minimal perfect coloring is monochromatic.

Theorem (T., 2021+)

Let $\mathcal{G}$ be a hypergraph. Then there is the minimal perfect coloring f such that every other perfect coloring of $\mathcal{G}$ is obtained from f by slitting of color classes.

Minimal coloring

Theorem (Weisfeiler, Leman, 1968)

Let G be a graph. Then there is the minimal perfect coloring f such that every other perfect coloring of G is obtained from f by splitting of color classes.

For regular graphs the minimal perfect coloring is monochromatic.

Theorem (T., 2021+)

Let $\mathcal{G}$ be a hypergraph. Then there is the minimal perfect coloring f such that every other perfect coloring of $\mathcal{G}$ is obtained from f by slitting of color classes.

The proof relies on the existence of the minimal perfect coloring for the bipartite representation of a hypergraph.

Coverings of hypergraphs

A hypergraph $\mathcal{G}$ is a covering of a hypergraph $\mathcal{H}$, if there exists a perfect coloring of $\mathcal{G}$ whose parameter matrix $\mathbb{S}$ is the full adjacency matrix of $\mathcal{H}$.

Coverings of hypergraphs

A hypergraph $\mathcal{G}$ is a covering of a hypergraph $\mathcal{H}$, if there exists a perfect coloring of $\mathcal{G}$ whose parameter matrix $\mathbb{S}$ is the full adjacency matrix of $\mathcal{H}$.

Equivalently: $\mathcal{G}$ is a k-covering of $\mathcal{H}$ if one get the hypergraph $\mathcal{H}$ uniting suitable groups of k vertices in $\mathcal{G}$ and preserving the adjacency between them.

Coverings of hypergraphs

A hypergraph $\mathcal{G}$ is a covering of a hypergraph $\mathcal{H}$, if there exists a perfect coloring of $\mathcal{G}$ whose parameter matrix $\mathbb{S}$ is the full adjacency matrix of $\mathcal{H}$.

Equivalently: $\mathcal{G}$ is a k-covering of $\mathcal{H}$ if one get the hypergraph $\mathcal{H}$ uniting suitable groups of k vertices in $\mathcal{G}$ and preserving the adjacency between them.

Theorem (T., 2021+)

If a hypergraph $\mathcal{G}$ is a covering of $\mathcal{H}$, then every eigenvalue of $\mathcal{H}$ is an eigenvalue of $\mathcal{G}$.

Coverings of hypergraphs

Theorem (T., 2021+)

Let $\mathcal{G}$ be a covering of a hypergraph $\mathcal{H}$. Then for every perfect coloring of $\mathcal{H}$ with the parameter matrix $\mathbb{S}$, there is a perfect coloring of $\mathcal{G}$ with the same parameter matrix $\mathbb{S}$.

Coverings of hypergraphs

Theorem (T., 2021+)

Let $\mathcal{G}$ be a covering of a hypergraph $\mathcal{H}$. Then for every perfect coloring of $\mathcal{H}$ with the parameter matrix $\mathbb{S}$, there is a perfect coloring of $\mathcal{G}$ with the same parameter matrix $\mathbb{S}$.

Theorem (Leighton, 1982)

Graphs H_1 and H_2 have the minimal perfect coloring with the same parameter matrix if and only if there exists a graph G covering both H_1 and H_2.

Coverings of hypergraphs

Theorem (T., 2021+)

Let $\mathcal{G}$ be a covering of a hypergraph $\mathcal{H}$. Then for every perfect coloring of $\mathcal{H}$ with the parameter matrix $\mathbb{S}$, there is a perfect coloring of $\mathcal{G}$ with the same parameter matrix $\mathbb{S}$.

Theorem (Leighton, 1982)

Graphs H_1 and H_2 have the minimal perfect coloring with the same parameter matrix if and only if there exists a graph G covering both H_1 and H_2.

Theorem (T., 2021+)

Hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ have the minimal perfect coloring with the same parameter matrix if and only if there exists a hypergraph $\mathcal{G}$ covering both $\mathcal{H}_1$ and $\mathcal{H}_2$.

Thank you for your attention!