On coverings and perfect colorings of hypergraphs

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Perfect coloring of graphs

A perfect $k$-coloring (equitable $k$-partition) is a function $f$ from the vertex set to colors $\{1, \ldots, k\}$ such that each vertex of color $i$ is adjacent to exactly $s_{i,j}$ vertices of color $j$. 

$S = (s_{i,j})$ is the parameter matrix of a perfect coloring.
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Eigenvalues and coverings of graphs

Adjacency matrix $M$ of a graph $G$: $m_{i,j} = 1$ if $(i,j)$ is an edge, $m_{i,j} = 0$ otherwise.
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Perfect coloring is a $(0, 1)$-matrix $P$, $p_{v,j} = 1 \iff f(v) = j$, such that

$$MP = PS.$$

Eigenvalues and eigenvectors of a graph $G$ are eigenvalues and eigenvectors of the adjacency matrix $M$. 
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A covering of a graph $H$ by a graph $G$ = a perfect coloring of $G$ with the parameter matrix equal to the adjacency matrix of $H$. 
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Our goal: develop the similar concepts for hypergraphs
Hypergraphs, incidence matrices, bipartite representation

\(G(X, W)\) is a hypergraph, 
\(X\) is the vertex set, \(|X| = n\), \(W\) is the hyperedge set, \(|W| = m\).
Hypergraphs, incidence matrices, bipartite representation

$\mathcal{G}(X, W)$ is a hypergraph,
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The incidence matrix $I$ of $\mathcal{G}$ is an $(n \times m)$-rectangular $(0, 1)$-matrix with a $(x, w)$-entry equals 1 $\iff x \in w$ in $\mathcal{G}$.
Hypergraphs, incidence matrices, bipartite representation

\( \mathcal{G}(X, W) \) is a hypergraph, 
X is the vertex set, \(|X| = n\), W is the hyperedge set, \(|W| = m\).

The incidence matrix \( I_\mathcal{G} \) of \( \mathcal{G} \) is an \((n \times m)\)-rectangular \((0, 1)\)-matrix with a \((x, w)\)-entry equals 1 \( \iff x \in w \) in \( \mathcal{G} \).

A degree \( deg(S) \) of \( S \subset X \) is the number of hyperedges, containing all vertices from \( S \).
A hypergraph is \textit{d-uniform} if each hyperedge consists of exactly \( d \) vertices.
Hypergraphs, incidence matrices, bipartite representation

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The bipartite representation \( G(X, W; E) \) of a hypergraph \( \mathcal{G}(X, W) \) is a bipartite graph, \( x \) is adjacent to \( w \) in \( G \) iff \( x \) is incident to \( w \) in \( \mathcal{G} \).
The adjacency matrix \( M_G \) of the bipartite representation is

\[
M_G = \begin{pmatrix} 0 & I \\ \|T & 0 \end{pmatrix}.
\]
Plain adjacency matrix of a hypergraph

Let $G$ be a hypergraph with the incidence matrix $\mathcal{I}$.

Simple graph is a 2-uniform hypergraph. The adjacency matrix of a graph is $M = \mathcal{I}^T - D$, where $D$ is the diagonal degree matrix.
Plain adjacency matrix of a hypergraph

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Simple graph is a 2-uniform hypergraph. The adjacency matrix of a graph is \( M = \mathbb{I}^T \mathbb{I} - D \), where \( D \) is the diagonal degree matrix.

Let the plain adjacency matrix of a hypergraph \( \mathcal{G} \) be the matrix \( M = \mathbb{I}^T \mathbb{I} - D \).

Plain eigenvalues and eigenvectors of a hypergraph are eigenvalues and eigenvectors of its plain adjacency matrix.
Plain adjacency matrix of a hypergraph

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Let the plain adjacency matrix of a hypergraph $G$ be the matrix $M = I^T I - D$.

Plain eigenvalues and eigenvectors of a hypergraph are eigenvalues and eigenvectors of its plain adjacency matrix.

Such approach to the adjacency matrix was used for studying Berge cycles, metric and expanding properties of hypergraphs.
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Another approach: the adjacency matrix of a $d$-uniform hypergraph is a $d$-dimensional matrix.
Multidimensional adjacency matrices

A *d*-dimensional matrix $A$ of order $n$ is an array $(a_{\alpha})$, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{1, \ldots, n\}$.

1. Combinatorial approach:
   The adjacency matrix $M$ of a $d$-uniform hypergraph $G$ on $n$ vertices is a $d$-dimensional $(0, 1)$-matrix of order $n$ with entries $m_{\alpha} = 1 \iff (\alpha_1, \ldots, \alpha_d)$ is a hyperedge in $G$.

2. Algebraic approach:
   For 2-dimensional matrices $B_1, \ldots, B_d$ of order $n$ define folding $C = [B_1, \ldots, B_d]$ to be the $d$-dimensional matrix of order $n$:
   $$c_{\alpha_1, \ldots, \alpha_d} = \sum_{i=1}^{n} b_{1, \alpha_1, i} \cdots b_{d, \alpha_d, i}.$$
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A \( d \)-dimensional matrix \( A \) of order \( n \) is an array \( (a_{\alpha}) \), \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( \alpha_i \in \{1, \ldots, n\} \).

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2. **Algebraic approach:** For 2-dimensional matrices \( B^1, \ldots, B^d \) of order \( n \) define folding \( C = [B^1, \ldots, B^d] \) to be the \( d \)-dimensional matrix of order \( n \):

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c_{\alpha_1, \ldots, \alpha_d} = \sum_{i=1}^{n} b^1_{\alpha_1,i} \cdots b^d_{\alpha_d,i}.
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The full adjacency matrix \( A \) of a \( d \)-uniform hypergraph \( G \) on \( n \) vertices is

\[
A = [\mathbb{1}, \ldots, \mathbb{1}].
\]
Proposition

Let $A$ be the full adjacency matrix of a $d$-uniform hypergraph. Then entries $a_\alpha$ are exactly the degrees of sets $S(\alpha) = \{\alpha_1, \ldots, \alpha_d\}$. 

Proposition

If $G$ is a $d$-uniform totally $(r_1, \ldots, r_{d-1})$-regular hypergraph, then $A = M + \sum_{t=1}^{d-1} r_t I_t$, where $I_t$ is a $d$-dimensional $(0, 1)$-matrix, whose unity entries indexed by $(\alpha_1, \ldots, \alpha_d)$ with exactly $t$ different components.
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We will say that a $d$-uniform hypergraph $G = (X, W)$ is totally $(r_1, \ldots, r_{d-1})$-regular if every $S \subset X$, $|S| = i$, has $\text{deg}(S) = r_i$. 
Full adjacency matrices and totally regular hypergraphs

Proposition

Let $\mathbf{A}$ be the full adjacency matrix of a $d$-uniform hypergraph. Then entries $a_{\alpha}$ are exactly the degrees of sets $S(\alpha) = \{\alpha_1, \ldots, \alpha_d\}$.

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If $\mathcal{G}$ is a $d$-uniform totally $(r_1, \ldots, r_{d-1})$-regular hypergraph, then

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Products of multidimensional matrices

The following multidimensional matrix products and eigenvalues were studied by L. Qi, L.H. Lim, S. Hu, C. Ling, J.-Yu Shao, Z. Huang, J. Cooper, A. Dutle, ...

Let $A$ be $d$-dimensional matrix of order $n$. Define product $\circ$:

If $v$ is a vector, then $A \circ v$ is a vector $u$:

$$u_{i_1,\ldots,i_{d-1}} = \sum_{i_1,\ldots,i_{d-1}=1}^n a_{j,i_1,\ldots,i_{d-1}} v_{i_1} \cdots v_{i_{d-1}}.$$  

If $P$ is a 2-dimensional matrix, then $A \circ P$ is a $d$-dimensional matrix $C$:

$$C_{j,k_1,\ldots,k_{d-1}} = \sum_{i_1,\ldots,i_{d-1}=1}^n a_{j,i_1,\ldots,i_{d-1}} p_{k_1,i_1,\ldots,i_{d-1}} \cdots p_{k_{d-1},i_{d-1}}.$$  

If $B$ is a $t$-dimensional matrix of order $n$, then $A \circ B$ is a similar $(d-1)(t-1) + 1$-dimensional matrix.
The following **multidimensional matrix products** and **eigenvalues** were studied by L. Qi, L.H. Lim, S. Hu, C. Ling, J.-Yu Shao, Z. Huang, J. Cooper, A. Dutle, ... 

Let $A$ be $d$-dimensional matrix of order $n$. Define **product** $\circ$:

- If $v$ is a vector, then $A \circ v$ is a vector $u$:
  
  $$u_j = \sum_{i_1,...,i_{d-1}=1}^{n} a_{j,i_1,...,i_{d-1}} v_{i_1} \cdots v_{i_{d-1}}.$$

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Coverings and colorings of hypergraphs

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- If $P$ is a 2-dimensional matrix, then $A \circ P$ is a $d$-dimensional matrix $C$:
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- If $B$ a $t$-dimensional matrix of order $n$, then $A \circ B$ is a similar $((d - 1)(t - 1) + 1)$-dimensional matrix.
Eigenvalues of hypergraphs

$\lambda$ is an eigenvalue and $\nu$ is the eigenvector of a $d$-dimensional matrix $A$ if

$$A \circ \nu = \lambda (I \circ \nu),$$

where $I = I_1$ is the $d$-dimensional identity matrix.
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Let $G$ be a $d$-uniform hypergraph. Eigenvalues and eigenvectors of $G$ are eigenvalues and eigenvectors of the adjacency matrix $M$. 

Theorem (T., 2021+)

Let $\nu$ be a plain eigenvector for $G$. Then $\nu$ is a full eigenvector for $G$; If $G$ is a totally regular hypergraph, then $\nu$ is an eigenvector for $G$.

In all cases, eigenvalues of one type can be counted from another.
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λ is an eigenvalue and \( v \) is the eigenvector of a \( d \)-dimensional matrix \( A \) if

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where \( I = I_1 \) is the \( d \)-dimensional identity matrix.

Let \( G \) be a \( d \)-uniform hypergraph.

Eigenvalues and eigenvectors of \( G \) are eigenvalues and eigenvectors of the adjacency matrix \( \mathbb{M} \).

Full eigenvalues and eigenvectors of \( G \) are eigenvalues and eigenvectors of the full adjacency matrix \( \mathbb{A} \).
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\( \lambda \) is an eigenvalue and \( \nu \) is the eigenvector of a \( d \)-dimensional matrix \( A \) if

\[ A \circ \nu = \lambda (I \circ \nu), \]

where \( I = I_1 \) is the \( d \)-dimensional identity matrix.

Let \( \mathcal{G} \) be a \( d \)-uniform hypergraph.

**Eigenvalues and eigenvectors** of \( \mathcal{G} \) are eigenvalues and eigenvectors of the adjacency matrix \( M \).

**Full eigenvalues and eigenvectors** of \( \mathcal{G} \) are eigenvalues and eigenvectors of the full adjacency matrix \( A \).

**Theorem (T., 2021+)**

Let \( \nu \) be a plain eigenvector for \( \mathcal{G} \). Then

- \( \nu \) is a full eigenvector for \( \mathcal{G} \);
- If \( \mathcal{G} \) is a totally regular hypergraph, then \( \nu \) is an eigenvector for \( \mathcal{G} \).

In all cases, eigenvalues of one type can be counted from another.
Perfect colorings of hypergraphs

Let $G = (X, W)$ be a $d$-uniform hypergraph with the incidence matrix $I$.

**Direct definition:** A function $f : X \rightarrow \{1, \ldots, k\}$ is a perfect $k$-coloring of $G$ if a coloring of a vertex uniquely defines the coloring of all incident hyperedges.

**Using bipartite representation:** A perfect $k$-coloring of $G$ is given by vertex $k$-coloring matrix $P$ and hyperedge coloring matrix $R$ satisfying

$$0 I I^T 0 = 0 P R 0 \quad S^T 0.

S$ is the HV-parameter matrix; $T$ is the VH-parameter matrix.

**Multidimensional definition:** A vertex $k$-coloring matrix $P$ defines a perfect coloring of $G$ if $A \circ P = P \circ S$.

Parameter matrix $S$ is a $d$-dimensional matrix of order $k$.

All three definitions are equivalent.
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(0 I I^T 0) (0 P R 0) = (0 P R 0) (0 S^T 0),
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I^T & 0
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0 & P \\
R & 0
\end{pmatrix} = 
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$$\mathcal{A} \circ P = P \circ \mathcal{S}.$$

Parameter matrix $\mathcal{S}$ is a $d$-dimensional matrix of order $k$. 
Perfect colorings of hypergraphs

Let $G = (X, W)$ be a $d$-uniform hypergraph with the incidence matrix $I$.

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$$
\left( \begin{array}{cc}
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\end{array} \right) \left( \begin{array}{cc}
0 & P \\
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$S$ is the HV-parameter matrix; $T$ is the VH-parameter matrix.

**Multidimensional definition:** A vertex $k$-coloring matrix $P$ defines a perfect coloring of $G$ if

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Parameter matrix $S$ is a $d$-dimensional matrix of order $k$.

All three definitions are equivalent.
Transversals in hypergraphs

Let $\mathcal{G} = (X, W)$ be a $d$-uniform $r$-regular hypergraph.
A $t$-transversal in $\mathcal{G}$ is a set $U \subseteq X$ such that each hyperedge $w$ contains exactly $t$ vertices from $U$. 

HV- and VH-parameter matrices:
$S = \begin{pmatrix} t & d - t \end{pmatrix}$; $T = \begin{pmatrix} r & r \end{pmatrix}$. 

The parameter matrix $S$:
$S_{\alpha} = r d \prod_{i=2}^{\alpha} t_{\alpha i} (d - t_{\alpha i})^{1 - \alpha i}$; $\alpha_i \in \{0, 1\}$.
Transversals in hypergraphs

Let \( \mathcal{G} = (X, W) \) be a \( d \)-uniform \( r \)-regular hypergraph. A \( t \)-transversal in \( \mathcal{G} \) is a set \( U \subseteq X \) such that each hyperedge \( w \) contains exactly \( t \) vertices from \( U \).

If \( U \) is a \( t \)-transversal in \( \mathcal{G} \), then \( U \) and \( X \setminus U \) are the color classes of a perfect 2-coloring of \( \mathcal{G} \).
Transversals in hypergraphs

Let $\mathcal{G} = (X, W)$ be a $d$-uniform $r$-regular hypergraph. A $t$-transversal in $\mathcal{G}$ is a set $U \subseteq X$ such that each hyperedge $w$ contains exactly $t$ vertices from $U$.

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$$S = \begin{pmatrix} t & d - t \end{pmatrix}, \quad T = \begin{pmatrix} r \\ r \end{pmatrix}.$$
Transversals in hypergraphs

Let $\mathcal{G} = (X, W)$ be a $d$-uniform $r$-regular hypergraph. A $t$-transversal in $\mathcal{G}$ is a set $U \subseteq X$ such that each hyperedge $w$ contains exactly $t$ vertices from $U$.

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HV- and VH-parameter matrices:

$$S = \begin{pmatrix} t & d - t \end{pmatrix}; \quad T = \begin{pmatrix} r & r \end{pmatrix}.$$ 

The parameter matrix $S$:

$$S_\alpha = r \prod_{i=2}^{d} t^{\alpha_i} (d - t)^{1 - \alpha_i}; \quad \alpha_i \in \{0, 1\}.$$
Let $G$ be a $d$-uniform hypergraph with the full adjacency matrix $A$. $P$ is a perfect $k$-coloring of $G$ if

$$A \circ P = P \circ S.$$ 

$S$ is the parameter matrix of the perfect coloring.
Parameter matrix of a hypergraph

Let $G$ be a $d$-uniform hypergraph with the full adjacency matrix $A$. $P$ is a perfect $k$-coloring of $G$ if

$$A \circ P = P \circ S.$$ 

$S$ is the parameter matrix of the perfect coloring.

**Theorem (T., 2021+)**

$S$ is the HV-parameter matrix and $T$ is the VH-parameter matrix of a perfect coloring of a hypergraph $G$ if and only if

$$S = [T, S^T, \ldots, S^T].$$
Properties of parameter matrices

In case of totally regular hypergraphs, perfect colorings can be defined as through the adjacency matrix, as the full adjacency matrix.

Proposition
If $G$ is a uniform totally regular hypergraph, $P$ is a vertex coloring matrix, then $A \circ P = P \circ S \iff M \circ P = P \circ T$.

Theorem (T., 2021+)
Let $P$ be a perfect coloring of a hypergraph $G$ with the parameter matrix $S$. If $\lambda$ and $v$ are eigenvalue and eigenvector of $S$, then $\lambda$ and $v$ are eigenvalue and eigenvector of the full adjacency matrix $A$.

Theorem (T., 2021+)
The parameter matrix $S$ of a perfect coloring of a hypergraph can be symmetrized by the vector of color densities.
Properties of parameter matrices

In case of **totally regular** hypergraphs, perfect colorings can be defined as through the **adjacency matrix**, as the **full adjacency matrix**.

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Theorem (Weisfeiler, Leman, 1968)

Let $G$ be a graph. Then there is the minimal perfect coloring $f$ such that every other perfect coloring of $G$ is obtained from $f$ by splitting of color classes.

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Coverings of hypergraphs

A hypergraph $G$ is a covering of a hypergraph $H$, if there exists a perfect coloring of $G$ whose parameter matrix $S$ is the full adjacency matrix of $H$. 

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Anna Taranenko  Coverings and colorings of hypergraphs  taa@math.nsc.ru
Coverings of hypergraphs

Theorem (T., 2021+)

Let $G$ be a covering of a hypergraph $H$. Then for every perfect coloring of $H$ with the parameter matrix $S$, there is a perfect coloring of $G$ with the same parameter matrix $S$. 
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Graphs $H_1$ and $H_2$ have the minimal perfect coloring with the same parameter matrix if and only if there exists a graph $G$ covering both $H_1$ and $H_2$. 
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Thank you for your attention!