On Modular Skew Lattices

and their coset structure

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arXiv: Costa, J P

inspired by Michael Kynion's 2009 talk in CAUL, Lisbon

Noncommutative structures within order structures, semigroups and universal algebra (MS - ID 67),
8th European Congress of Mathematics

Motivation

Computational Topology has foundations on persistent modules, i.e., graded modules over the polynomial ring $k[t]$ over a coefficient field k - arXiv:1302.2015

Modular law

$$a \leq b \text{ implies } a \vee (x \wedge b) = (a \vee x) \wedge b \text{ for every } x$$

Example. Subspaces of a vector space and, in general, submodules of a module over a ring.

Theor [Dedekind]. A lattice is modular iff it does not contain the pentagon as a sublattice.

Cor [Dedekind, 1900]. Every maximal chain in a finite dimensional modular lattice has the same legth.

Skew Lattices

A *skew lattice* is a set S with binary operations $\wedge$ and $\vee$ that are both idempotent and associative, satisfying the absorption laws $x \wedge (x \vee y) = x = (y \vee x) \wedge x$ and their duals.
Skew lattices are double regular bands of semigroups, considering the regular band reducts $(S, \wedge)$ and $(S, \vee)$. The order relations are:

the *natural partial order* by $x \leq y$ iff $y \wedge x \wedge y = x$;

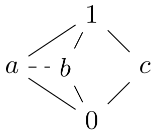
the *natural preorder* by $x \preceq y$ iff $x \wedge y \wedge x = x$;

the *natural equivalence* by $x\mathcal{D}y$ iff $x \preceq y$ and $y \preceq x$.

If S is a skew lattice, D in S is a congruence and $S/\mathcal{D}$ is a lattice. Its *right-handed* reduct $\mathcal{R}$ is defined by $x\mathcal{R}y$ iff $y \wedge x = x = x \vee y$ and $x \wedge y = y = y \vee x$; The left-handed version $\mathcal{L}$ is similar.

Orders and Ideals

An *admissible Hasse diagram* of a skew lattice is a Hasse diagram for the natural partial order (represented by full edges) together with an indication of all $\mathcal{D}$-congruent elements (represented by dashed edges).



$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	b	0	a
b	0	a	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	a	1	1
b	b	b	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

Figure: *Caley tables and admissible Hasse diagram of the righ-handed* $\mathbf{NC}_5^R$.

Skew Lattices in Rings of Matrices

Let F be a field such that $char F \neq 2$. Consider the ring $M_n(F)$ and let $E(M_n(F))$ the set of idempotent elements in $M_n(F)$. $S \subset E(M_n(F))$ is a right-handed skew lattice in $M_n(F)$ with operations

$$x \wedge y = xy \text{ and } x \vee y = x \circ y = x + y - xy.$$

Let S consist of all matrices of the form

$$\begin{bmatrix} 0 & a_1 & a_2 & \dots & a_k & c \\ 0 & e_1 & 0 & \dots & 0 & b_1 \\ 0 & 0 & e_2 & \dots & 0 & b_2 \\ 0 & 0 & 0 & \dots & e_k & b_k \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

where $e_i \in \{0, 1\}$, $a_i = a_i e_i$, $b_j = e_j b_j$ and $c = \sum_i a_i b_i$. Thus, $a_i = 0$ if $e_i = 0$ and $b_j = 0$ if $e_j = 0$. $(S, ., \circ)$ is a skew lattice.

Quasimodular Skew Lattices

A skew lattice $\mathbf{S}$ is *quasimodular* [*quasidistributive*] if $S/\mathcal{D}$ is a modular [distributive] lattice.

Prop..[Kinyon 2009] Let $\mathbf{S}$ be a skew lattice. Then,

(i) $\mathbf{S}$ is quasimodular iff it satisfies $x \wedge (y \vee (z \wedge (y \vee x \vee y) \wedge z) \vee y) \wedge x = x \wedge (y \vee z \vee y) \wedge x$;

(ii) $\mathbf{S}$ is quasidistributive iff it satisfies $x \wedge (y \vee (z \wedge x \wedge z) \vee y) \wedge x = x \wedge (y \vee z \vee y) \wedge x$.

(iiii) quasimodular iff it does not contain N5 as a subalgebra;

(iv) quasidistributive iff it contains neither M3 nor N5 as subalgebras.

Quasicancellative and quasidistributive are equivalent notions.
Quasidistributive implies quasimodular.

Biconditionally Distributive Skew Lattices

A skew lattice $\mathbf{S}$ is *biconditionally distributive* if *BCD1* holds for any particular $x, y, z \in S$ iff *BCD2* does. [Leech 2011]

$$x \vee ((x \vee y) \wedge z) \vee x = x \vee (y \wedge (z \vee x)) \vee x \qquad \text{(BCD1)}$$

$$x \wedge ((x \wedge y) \vee z) \wedge x = x \wedge (y \vee (z \wedge x)) \wedge x \qquad \text{(BCD2)}$$

- A (commutative) lattice is biconditionally distributive if and only if it is modular.
- Biconditional distributivity is a characterisation of quasi-modularity for skew lattices

Distributive and Cancellative Skew Lattices

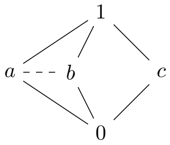
$\mathbf{S}$ is **cancellative** if for all $x, y, z \in S$,
$x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ imply $y = z$ and
$x \vee y = z \vee y$ and $x \wedge y = z \wedge y$ imply $x = z$.

$\mathbf{S}$ is **distributive** if for all $x, y, z \in S$,
$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and
$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$.

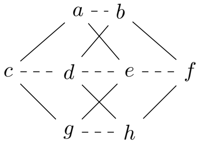
Unlike what happens with lattices,
Distributivity and cancellation are independent properties.

Varieties of Skew Lattices

Table: Distributivity and cancellation are independent properties.



Distributive but not cancellative



Cancellative but not distributive

Skew modular lattices MUST be quasimodular and MUST exist skew modular lattices that are not cancellative and others that are not distributive: the examples above.

Lower/upper Skew Modularity

[Distributivity: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$.]

Let $\mathbf{S}$ be a skew lattice. Then,

(i) $\mathbf{S}$ is *lower modular* if
$x \wedge (y \vee (x \wedge z \wedge x) \vee y) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x) \vee (x \wedge y \wedge x)$;

(ii) $\mathbf{S}$ is *upper modular* if
$x \vee (y \wedge (x \vee z \vee x) \wedge y) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x) \wedge (x \vee y \vee x)$;

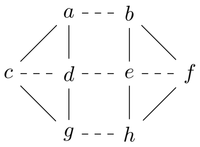
(iii) $\mathbf{S}$ is *modular* if it is both upper and lower modular.

[Kinyon 2009] The modular laws are independent in skew lattices, but equivalent in symmetric skew lattices.

Linear modularity

A skew lattice $\mathbf{S}$ is *linearly modular (distributive)* if every skew subchain is modular (distributive).

Example[Kinyon 2009] The smallest (right-handed) quasimodular skew chain not (linearly) modular. Let's call it *CR7*.



It is cancellative and, thus, cancellativity does not imply (linear) modularity. But distributivity does.

Linear modularity

[Kinyon 2009] Let $\mathbf{S}$ be a quasimodular, linearly modular skew lattice.

- if $\mathbf{S}$ is simply cancellative then $\mathbf{S}$ is modular;
- if $\mathbf{S}$ is symmetric then $\mathbf{S}$ is modular.

Remark There are modular (and, thus, quasimodular and linearly modular) skew lattices that are not simply cancellative nor symmetric. Distributive skew lattices are modular, but the converse does not hold. Skew lattices of matrices in rings are modular (but also distributive and symmetric).

- A simply cancellative skew lattice is distributive (modular?) iff it is linearly distributive (modular?).

Normal Skew Lattices

The center of a skew lattice **S** is subalgebra formed by the union of all its singleton D-classes. In particular, **S** is a lattice if either $\vee$ or $\wedge$ is commutative.

A skew lattice is normal if $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$, and is

normal iff $S \wedge x \wedge S$ is a sublattice, for each $x \in S$.

conormal iff $S \vee x \vee S$ is a sublattice, for each $x \in S$.

- The center of a skew lattice is always a normal skew lattice.
- A normal skew lattice is modular iff it is quasi-modular.

Categorical and Strictly Categorical

Given $A > B$ and cosets $B \vee a \vee B$ in A and $A \wedge b \wedge A$ in B a coset bijection is:

$$x \in B \vee a \vee B \text{ maps to } y \in A \wedge b \wedge A \text{ if and only if } x \geq y;$$

Categorical skew lattices form a variety (Leech & Kinyon 2012)

- Nonempty composites of coset bijections are coset bijections

S_{37}. $x \wedge [(x \wedge z \wedge x) \vee y \vee (x \wedge z \wedge x)] \wedge x = x \wedge [(z \wedge x \wedge z) \vee y \vee (z \wedge x \wedge z)] \wedge x;$

S_{38}. $x \vee [(x \vee z \vee x) \wedge y \wedge (x \vee z \vee x)] \vee x = x \vee [(z \vee x \vee z) \wedge y \wedge (z \vee x \vee z)] \vee x.$

Strictly categorical skew lattices form a variety (Leech & Kinyon 2012)

- The compositions of coset bijections are never empty.

S_{39}. $x \vee (y \wedge z \wedge u \wedge y) \vee x = x \vee (y \wedge u \wedge z \wedge y) \vee x;$

S_{40}. $x \wedge (y \vee z \vee u \vee y) \wedge x = x \wedge (y \vee u \vee z \vee y) \wedge x.$

Categorical and Strictly Categorical

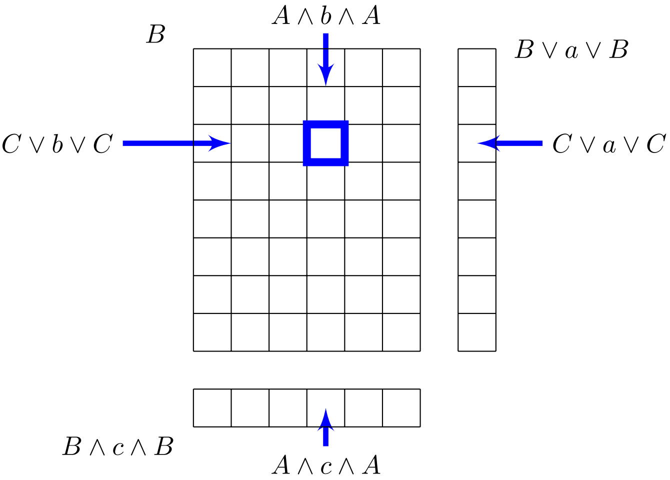


Figure: Diagram representation of the bijections in s strictly categorical skew lattice where all the squares are nonempty.

Categorical and Strictly Categorical

- *CR7* is quasimodular but not modular neither categorical.
- $1 > a - b > 0$ is modular but not strictly categorical.
- A strictly categorical skew lattice is distributive iff it is quasidistributive.
- Strictly categorical, quasidistributive skew lattices are both distributive and simply cancellative. They are cancellative precisely when they are also symmetric.

(Classical) Ideals of a Skew Lattice

A nonempty subset I of a skew lattice S is an **ideal** of S if for all $x, y \in I$, $x \vee y \in I$ and one of the following equivalent statements hold:

$i)$ for all $x \in I$ and $y \in S$, $x \preceq y$ implies $x \in I$;

$ii)$ for all $x \in I$ and $y \in S$, $y \wedge x, x \wedge y \in I$;

$iii)$ for all $x \in I$ and $y \in S$, $x \wedge y \wedge x \in I$.

The concept of *filter* is defined similarly.

An Example ...

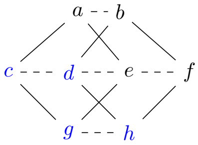


Figure: (Classical) Ideal of a Skew Lattice

Principal (Classic) Ideals

[Leech & Bignall 1995] Let **S** be a skew lattice and $x \in S$.

$$(x] = S \wedge x \wedge S = \{\, y \in S : y \preceq x \,\} = (x] = \bigcup(\mathcal{D}_x]$$

is the principal ideal determined by $x \in S$.
[Leech & Spinks 2008] If $S/\mathcal{D}$ is finite then all of its ideals are principal with the lattice of ideals being naturally isomorphic to $S/\mathcal{D}$.

- If $S/\mathcal{D}$ is finite and quasimodular, then all the maximal chains of S have the same size.
- [Dilworth 1950] Every $k+1$ element skew chain of a poset A be dependent while at least one set of k elements is independent. Then, A is a disjoint union of k chains.
- [Mirsky 1971] The size of the largest chain in a partial order (if finite) equals the smallest number of antichains into which the order may be partitioned.

Skew Ideals

A nonempty proper subset I of S is a **skew ideal** of **S** if for all $x, y \in I$, $x \vee y, x \vee y \in I$, and if one of the following holds:

(i) for all $x \in I$ and $y \in S$, $x \geq y$ implies $y \in I$.

(ii) for all $x \in I$ and $y \in S$, $x \wedge y \wedge x \in I$.

The concept of *skew filter* is defined similarly.

Another Example ...

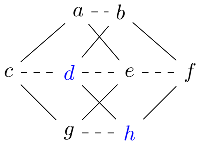


Figure: Skew Ideal of a Skew Lattice

Principal Skew Ideals

[Pita Costa 2012] Let **S** be a skew lattice and $x \in S$.

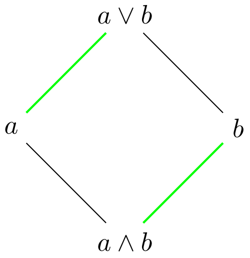
$$(x]^* = x \wedge S \wedge x = \{\, y \in S : y \leq x \,\}$$

is the principal weak ideal determined by $x \in S$. For all $x \in S$,

- $(x]^* \cap \mathcal{D}_x = \{\, x \,\}$;
- $\mathcal{D}_x \leq \mathcal{D}_a \Rightarrow (x]^* \cap \mathcal{D}_a \neq \emptyset$;
- $(x]^*$ it's a transversal of $(x] \cap \mathcal{D}_y$, when $\mathcal{D}_y \leq \mathcal{D}_x$.
- if S is a normal skew lattice, then S is quasi-modular and, for all $x \in S$, $(x]^*$ is modular.

Diamond isomorphism theorem

Lattice modularity determines a natural generalization of the *diamond isomorphism theorem* describing the isomorphism between $[a \wedge b, b]$ and $[a, a \vee b]$ using the maps $f : (a \vee b)/a \to b/(a \wedge b)$, and $g : b/(a \wedge b) \to (a \vee b)/a$, $y \mapsto a \vee y$.



There is a natural generalization that can be defined based on skew ideals and (classical) ideals.

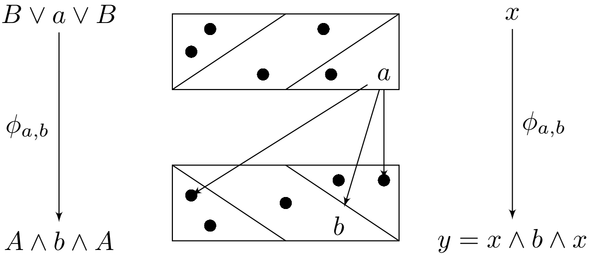
Coset Structure Theorem (Leech 1993)

Let $\mathbf{S}$ be a skew lattice with comparable $\mathcal{D}$-classes $A > B$.

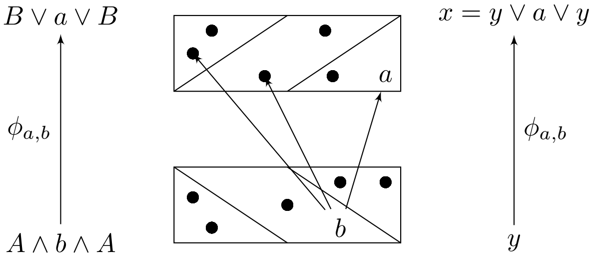
- B is partitioned by the cosets of A in B and, dually, A is partitioned by the cosets of B in A;
- Given cosets $B \vee a \vee B$ in A and $A \wedge b \wedge A$ in B a natural bijection of cosets is given by:

  $x \in B \vee a \vee B$ maps to $y \in A \wedge b \wedge A$ if and only if $x \geq y$;
- For all $x \in A$, $x \wedge B \wedge x$ is a transversal of cosets of A in B;
- The operations $\wedge$ and $\vee$ on $A \cup B$ are determined jointly by the coset bijections and the rectangular structure of each $\mathcal{D}$-class.

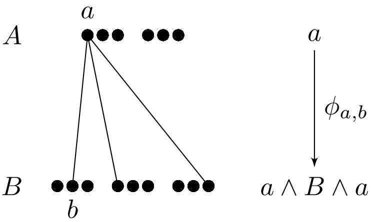
Coset Structure



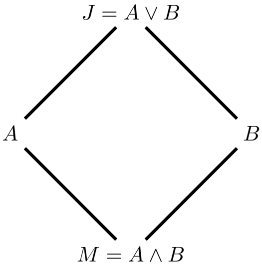
Coset Structure



Transversal of Cosets



Coset Laws for Cancellative Skew Lattices



Given a skew diamond $\{ J > A, B > M \}$, for every $a \in A$ there exists $b \in B$ such that $a \vee b = b \vee a$ in J and $a \wedge b = b \wedge a$ in M.

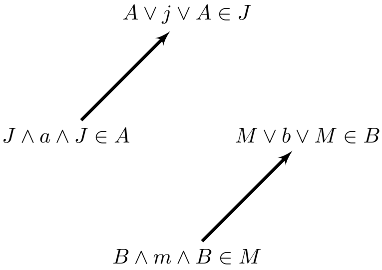
Moreover, $J = \{a \vee b \mid a \in A, b \in B \text{ and } a \vee b = b \vee a\}$ and $M = \{ a \wedge b \mid a \in A, b \in B \text{ and } a \wedge b = b \wedge a \}$.

Coset Laws for Cancellative Skew Lattices

A quasi-distributive, symmetric skew lattice $\mathbf{S}$ is cancellative **iff** for all $\{ J > A, B > M \}$ in $\mathbf{S}$ and any $x, x' \in A$, the following hold:

i) $M \vee x \vee M = M \vee x' \vee M$ iff $B \vee x \vee B = B \vee x' \vee B$;

ii) $B \wedge x \wedge B = B \wedge x' \wedge B$ iff $J \wedge x \wedge J = J \wedge x' \wedge J$.



Coset Laws for Distributive Skew Lattices

[Kinyon-Leech-PitaCosta 2015, 2018] A skew chain $A > B > C$ is distributive iff for all $a \in A$, $b \in B$ and $c \in C$ such that $a > c$,

$$A \wedge (b \vee c \vee b) \wedge A = A \vee (b \wedge c \wedge b) \vee A$$

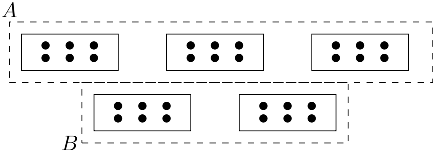
and the same is valid for all b in a common AC-component of B and is the unique element y in that component such that $a > y > c$.

(AC-*component* in B is a maximal AC-connected subset of B)

This is equivalent to: for all AC-components B' in B, $A > B' > C$ is strictly categorical.

Index and Order of a Coset

$\mathbf{S}$ skew lattice with comparable $\mathcal{D}$-classes $A > B$. The **index** of B in A is the number of all B-cosets in A, $[B : A] = |a \wedge B \wedge a|$, for any $a \in A$. The index of A in B is defined similarly.



All A-cosets in B and all B-cosets in A have a common size $\omega[A, B]$ or, equivalently, $\omega[B, A]$. A is finite if and only if $[A : B]$ and $\omega[A, B]$ are finite and, in that case,

$$|A| = [A : B].\omega[A, B].$$

Likewise, $|B| = [B : A].\omega[A, B]$ with similar remarks on the finitude of B.

Index Theorems for Finite Skew Lattices

Given a skew chain $A > B > C$ in a skew lattice $\mathbf{S}$ with both A and C finite,

- $A > B > C$ is strictly categorical iff
$$|B| = \frac{\omega[A,B]\omega[B,C]}{\omega[A,C]}$$
- $A > B > C$ is distributive iff
$$|B| = n\frac{\omega[A,B]\omega[B,C]}{\omega[A,C]}.$$

where n is the number of AC components of B.

Open Problems

- Can we identify new properties specific to modular skew lattices from their coset structure?
- Can the coset law-like identities that characterize modular skew lattices help us to compare them with other varieties of skew lattices?
- Can we elaborate combinatorial rules on index theorems for modular skew lattices?