Periodic random tilings and non-Hermitian orthogonality

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The talk is based on

- **M. Duits and A.B.J. Kuijlaars,**

- **C. Charlier, M. Duits, A.B.J. Kuijlaars, and J. Lenells,**

- **Alan Groot and Arno B.J. Kuijlaars,**
  Matrix valued orthogonal polynomials related to hexagon tilings, preprint arXiv:2104.14822
1. Tiling problems:

Hexagon and Aztec diamond
1. Lozenge tiling of a hexagon

three types of lozenges
1 Arctic circle phenomenon
1 Domino tiling of Aztec diamond

- Tiling with $2 \times 1$ and $1 \times 2$ rectangles (dominos)
- Four types of dominos
Large random tiling: Arctic circle

Deterministic pattern near corners
Solid region or frozen region

Disorder in the middle
Liquid region

Boundary curve
Arctic circle
1 Some History

Number of domino tilings of Aztec diamond is $2^{N(N+1)/2}$
Elkies, Kuperberg, Larsen, Propp (1992)

Arctic circle phenomenon Jockush, Propp, Shor (1995)

Fluctuations around Arctic circle and connection to random matrix theory (Tracy-Widom distribution) Johansson (2002)

Arctic circle for hexagon tilings

- Johansson uses Krawtchouk polynomials
- Baik et al. use Hahn polynomials
2. Non-intersecting paths
Periodic random tilings and non-Hermitian orthogonality
Periodic random tilings and non-Hermitian orthogonality
2 Non-intersecting paths on a graph

Paths fit on a graph and give rise to multi level particle system
The multi-level particle system is determinantal on discrete state space $\mathcal{X} = \{0, 1, \ldots, L\} \times \mathbb{Z}$.

- There is correlation kernel $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ with property that for finite $\mathcal{A} \subset \mathcal{X}$

\[
\det [K(\vec{x}, \vec{y})]_{\vec{x}, \vec{y} \in \mathcal{A}} = \text{Prob} \begin{cases} 
\text{There is a particle at each } \vec{x} = (m, x) \in \mathcal{A} 
\end{cases}
\]

- Eynard Mehta (1998) give sum formula for the kernel in terms of transition matrices

\[
T_{m', m}(x, y) = \begin{cases} 
\# \text{paths on the lattice from } (m', x) \text{ to } (m, y) 
\end{cases}
\]

- The formula also works in a weighted setting.
2 Kernel for hexagon of size $N \times M \times (L - M)$

**Theorem (Duits K (2021) – very special case)**

Correlation kernel has **double contour integral formula**

\[
\begin{align*}
- \frac{Xm' > m}{2\pi i} \int_{\gamma} (z + 1)^{m'} - m \frac{z y - x}{z} d\zeta \\
+ \frac{1}{(2\pi i)^2} \int_{\gamma} \int_{\gamma} (w + 1)^{L-m'} R_N(w, z)(z + 1)^m \frac{w^y}{z^{x}w^{M+N}} \frac{dzdw}{z}
\end{align*}
\]

where \( R_N(w, z) = \sum_{k=0}^{N-1} \frac{p_k(w)p_k(z)}{h_k} \) is the **reproducing kernel** for orthogonal polynomials on a contour \( \gamma \) going around 0

\[
\frac{1}{2\pi i} \int_{\gamma} p_k(z)p_j(z) \frac{(z + 1)^L}{z^{M+N}} dz = h_k \delta_{k,j}
\]
Non Hermitian orthogonality on a contour in the complex plane.

The orthogonal polynomials are Jacobi polynomials

\[ p_k(z) \propto P_k^{(-M, -N, L)}(2z + 1) \]

with one negative parameter (!)
2 Non Hermitian orthogonality

\[
\frac{1}{2\pi i} \oint_{\gamma} p_k(z)p_j(z) \frac{(z + 1)^L}{z^{M+N}} \, dz = h_k \delta_{k,j}
\]

- **Non Hermitian orthogonality** on a contour in the complex plane.
- The orthogonal polynomials are **Jacobi polynomials**

\[
p_k(z) \propto P_k^{(-M-N,L)}(2z + 1)
\]

with one negative parameter (!)

- Similar formula applies to the **Aztec diamond**, but with Jacobi polynomials

\[
p_k(z) \propto P_k^{(-N,N)}(z)
\]
3. Weighted tilings
3 Weighted tilings

A weighting on tiles produces a weight on tilings $\mathcal{T}$

$$W(\mathcal{T}) = \prod_{T \in \mathcal{T}} w(T)$$

Probability of a tiling is

$$\text{Prob}(\mathcal{T}) = \frac{W(\mathcal{T})}{Z}, \quad Z = \sum_{\mathcal{T}'} W(\mathcal{T}')$$
3 Constant weights per column (example)

Weights depend on column

\[ w_{\square}(x, y) = \alpha_x , \]
\[ w_{\triangle}(x, y) = 1 , \]
\[ w_{\Box}(x, y) = 1 \]

Transition matrix

\[ T_m(x, y) = \begin{cases} 
\alpha_m & \text{if } y = x \\
1 & \text{if } y = x + 1 \\
0 & \text{otherwise}
\end{cases} \]
3 Constant weights per column (example)

Weights depend on column

\[ w(x, y) = \alpha_x, \]
\[ w(x, y) = 1, \]
\[ w(x, y) = 1 \]

Transition matrix

\[ T_m(x, y) = \begin{cases} 
\alpha_m & \text{if } y = x \\
1 & \text{if } y = x + 1 \\
0 & \text{otherwise}
\end{cases} \]

It is Toeplitz matrix with symbol \( \varphi_m(z) = z + \alpha_m \)
3 Correlation kernel (double contour part only)

Theorem

Correlation kernel at the \(m\)th level

\[
\frac{1}{(2\pi i)^2} \oint_{\gamma} \oint_{\gamma} \left( \prod_{j=m+1}^{L} \varphi_j(w) \right) R_N(w, z) \left( \prod_{j=1}^{m} \varphi_j(z) \right) \frac{w^y}{z^x w^{M+N}} \frac{dzdw}{z}
\]

where \(R_N(w, z) = \sum_{k=0}^{N-1} \frac{p_k(w)p_k(z)}{h_k}\) is the reproducing kernel for orthogonal polynomials on a contour \(\gamma\) going around 0

\[
\frac{1}{2\pi i} \oint_{\gamma} p_k(z)p_j(z) \frac{\prod_{j=1}^{L} \varphi_j(z)}{z^{M+N}} dz = h_k \delta_{k,j}
\]
3 Two periodic parameters

Suppose \( \alpha_m = \begin{cases} 
1 & \text{if } m \text{ is even} \\
\alpha & \text{if } m \text{ is odd}
\end{cases} \)

- Orthogonality weight is (for \( N = M = L - M \))

\[
\frac{(z + 1)^N(z + \alpha)^N}{z^{2N}}
\]

- This model has a phase transition in large \( N \) limit

Charlier, Duits, K, Lenells (2020)
Small parameter $0 < \alpha < 1/9$
Larger parameter $1/9 < \alpha < 1$

Phase transition at $\alpha = 1/9$

- Asymptotic analysis of the OP with Riemann-Hilbert problem and steepest descent analysis of double integral
4. Weightings that are periodic in vertical direction
4 Periodicity in vertical direction (example)

\[ w_\square(x, y) = \begin{cases} 
\alpha_x, & \text{if } y \text{ is even} \\
\beta_x, & \text{if } y \text{ is odd}
\end{cases} \]

\[ w_\triangle(x, y) = 1, \quad w_\Diamond(x, y) = 1 \]

Transition matrix is block Toeplitz

\[
T_m(x, y) = \begin{cases} 
\alpha_m & \text{if } y = x \text{ even} \\
\beta_m & \text{if } y = x \text{ odd} \\
1 & \text{if } y = x + 1 \\
0 & \text{otherwise}
\end{cases}
\]
4 Periodicity in vertical direction (example)

\[ w(x, y) = \begin{cases} 
\alpha_x, & \text{if } y \text{ is even} \\
\beta_x, & \text{if } y \text{ is odd} 
\end{cases} \]

\[ w(x, y) = 1, \quad w(x, y) = 1 \]

Transition matrix is block Toeplitz

\[ T_m(x, y) = \begin{cases} 
\alpha_m & \text{if } y = x \text{ even} \\
\beta_m & \text{if } y = x \text{ odd} \\
1 & \text{if } y = x + 1 \\
0 & \text{otherwise} 
\end{cases} \]

Block symbol

\[ \Phi_m(z) = \begin{pmatrix} 
\alpha_m & 1 \\
z & \beta_m 
\end{pmatrix} \]
4 Correlation kernel (double contour part only)

Theorem

Correlation kernel at the $m$th level (for $2N \times 2M \times L - 2M$ hexagon) are entries of

$$\frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma \left( \prod_{j=m+1}^{L} \Phi_j(w) \right) R_N(w, z) \left( \prod_{j=1}^{m} \Phi_j(z) \right) \frac{w^y}{z^{x} w^{M+N}} \frac{dzdw}{z}$$

where $R_N(w, z) = \sum_{k=0}^{N-1} P_k^T(w) H_k^{-1} P_k(z)$ is the reproducing kernel for matrix valued orthogonal polynomials on a contour $\gamma$ going around 0

$$\frac{1}{2\pi i} \oint_\gamma P_k(z) \frac{\prod_{j=1}^{L} \Phi_j(z)}{z^{M+N}} P_j(z)^T \, dz = H_k \delta_{k,j}$$
4 Comment

- The theorem extends to transition matrices with block Toeplitz structure of any periodicity.
5. Matrix valued orthogonal polynomials (MVOP)
Matrix valued orthogonal polynomials (MVOP)

\[ \frac{1}{2\pi i} \oint_{\gamma} P_k(z)W(z)P_j^T(z)dx = H_j \delta_{j,k}, \quad \det H_j \neq 0 \]

- **$W(z)$** is $p \times p$ matrix for every $z$
- **$P_k$** is matrix valued polynomial

\[ P_k(x) = C_0 x^k + C_1 x^{k-1} + \cdots, \quad C_i \text{ is } p \times p \text{ matrix.} \]

- Integral is taken entry-wise.
5 Matrix valued orthogonal polynomials (MVOP)

\[
\frac{1}{2\pi i} \oint_{\gamma} P_k(z) W(z) P_j^T(z) \, dx = H_j \delta_{j,k}, \quad \det H_j \neq 0
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- \( W(z) \) is \( p \times p \) matrix for every \( z \)
- \( P_k \) is matrix valued polynomial

\[
P_k(x) = C_0 x^k + C_1 x^{k-1} + \cdots, \quad C_i \text{ is } p \times p \text{ matrix.}
\]

- Integral is taken entry-wise.

**Questions** on existence and uniqueness, recurrence relations, generating functions, differential equations, ... 

- **Examples and Applications**: do MVOP appear in "real life" problems?
6. Two periodic Aztec diamond
6 Random tiling with uniform measure

- Deterministic pattern near corners
- Solid region or frozen region
- Disorder in the middle
- Liquid region
- Boundary curve Arctic circle
6 Aztec diamond; two-periodic weighting

A new phase within the liquid region:

gas region
(smooth region)

Chhita, Johansson (2016)
Beffara, Chhita, Johansson (2018)
6 Paths in the Aztec diamond

Line segments on West, East and South dominos

North

West

East

South

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6 Transformations and extension; particle system

- Rotate the Aztec diamond
- Extend the tiling to a double Aztec diamond
- Put particles on the paths
- Particles are a determinantal point process
6 Non-intersecting paths on a weighted graph

- Apply affine transformation

Two periodic weighting

- Bernoulli step with weight $\alpha$ or $\beta = \alpha^{-1}$

- Steps down plus horizontal step have weight 1
6 Symbols and weight

Block symbols are \[
\begin{pmatrix}
\alpha & \alpha \\
\beta z & \beta
\end{pmatrix}
\] and \[
\frac{1}{z - 1} \begin{pmatrix}
z & 1 \\
z & z
\end{pmatrix}
\]
6 Symbols and weight

Block symbols are \(
\begin{pmatrix}
\alpha & \alpha \\
\beta z & \beta
\end{pmatrix}
\) and \(
\frac{1}{z - 1}
\begin{pmatrix}
z & 1 \\
z & z
\end{pmatrix}
\)

Weight matrix is \(W^N\) for Aztec diamond of size \(2N\), where

\[
W(z) = \frac{1}{z(z - 1)^2} \begin{pmatrix}
\alpha & \alpha \\
\beta z & \beta
\end{pmatrix} \begin{pmatrix}
z & 1 \\
z & z
\end{pmatrix} \begin{pmatrix}
\alpha & \alpha \\
\beta z & \beta
\end{pmatrix} \begin{pmatrix}
z & 1 \\
z & z
\end{pmatrix}
\]

\[
= \frac{1}{(z - 1)^2} \begin{pmatrix}
(z + 1)^2 + 4\alpha^2 z & 2\alpha(\alpha + \beta)(z + 1) \\
2\beta(\alpha + \beta)z(z + 1) & (z + 1)^2 + 4\beta^2 z
\end{pmatrix}
\]
6 MVOP

MVOP of degree $N$ is explicit if $N$ is even

$$P_N(z) = (z - 1)^N W(\infty)^{N/2} W^{-N/2}(z)$$

- The double contour integral for the correlation kernel simplifies considerably
- Different approach is due to Berggren-Duits (2019)
6 MVOP

MVOP of degree $N$ is explicit if $N$ is even

\[
P_N(z) = (z - 1)^N W(\infty)^{N/2} W^{-N/2}(z)
\]

- The double contour integral for the correlation kernel simplifies considerably
- Different approach is due to Berggren-Duits (2019)
- What remains is saddle point analysis of the double contour integral.
- There are four saddle points (depending on position in the Aztec diamond) that "live" on two-sheeted spectral curve

\[
y^2 = z \left( z + \alpha^2 \right) \left( z + \beta^2 \right)
\]
6 Solid phase

- At least two saddles are in $[-\alpha^2, -\beta^2]$
- Other saddles $s_1$ and $s_2$ are in $[0, \infty)$ \implies solid phase
6 Liquid phase

Saddles $s_1$ and $s_2$ are not real $\implies$ liquid phase
6 Gas phase

**All saddles are in** \([-\alpha^2, -\beta^2]\) \(\Rightarrow\) **Gas phase**
6 Phase diagram

Thank you for your attention