Asymptotic consensus in the Hegselmann-Krause model with finite speed of information propagation

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The Hegselmann-Krause model (2002)

- Prototypical model for opinion dynamics: agents adapt their opinions to others’, with confidence depending on the difference in opinions.

- For $i = 1, \ldots, N$, $x_i = x_i(t) \in \mathbb{R}^d$ subject to

$$\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} \psi(|x_i - x_j|)(x_j - x_i)$$

The influence function $\psi \geq 0$ bounded, typically nonincreasing.

- For $\psi > 0$: convergence to global consensus

$$\lim_{t \to \infty} x_i = \bar{x} \quad \text{for all } i = 1, \ldots, N,$$

with

$$\bar{x} = \frac{1}{N} \sum_{i=0}^{N} x_i(0)$$

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Consensus with Finite Speed Propagation
Finite propagation speed

- Speed of light $c > 0$

- Agent located at $x_i = x_i(t)$ at time $t > 0$ observes the position of the agent $x_j$ at time $t - \tau_{ij}$, where $\tau_{ij}$ solves

  $$c\tau_{ij}(t) = |x_i(t) - x_j(t - \tau_{ij}(t))|$$

- Unique solvability guaranteed iff

  $$|\dot{x}_j(t)| \leq s \quad \text{for all } t \in \mathbb{R}$$

  with

  $$s < c$$

- Introduce the notation

  $$\tilde{x}_j^i := x_j(t - \tau_{ij}(t))$$

  ... information about position of $j$ received by $i$ at time $t$. 

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Consensus with Finite Speed Propagation
Finite propagation speed
We study the system

\[ \dot{x}_i = \frac{1}{N-1} \sum_{j=1}^{N} \psi(|\tilde{x}^i_j - x_i|) (\tilde{x}^i_j - x_i) \]

with \( \tilde{x}^i_j := x_j(t - \tau_{ij}(t)) \) unique solution of

\[ c\tau_{ij}(t) = |x_i(t) - x_j(t - \tau_{ij}(t))| \]

Subject to the \( s \)-Lipschitz continuous initial datum

\[ x_i(t) = x_i^0(t) \quad \text{for } i = 1, \ldots, N, \quad t \leq 0 \]

Central speed limit assumption:

\[ s := \sup_{r > 0} \psi(r)r < c \]

then \( |\dot{x}_i| \leq s \) for all \( i = 1, \ldots, N \).
Results I: Well posedness

- **Local** existence and uniqueness of solutions: Based on an adaptation of Picard-Lindelöf theorem - contraction in the space of $s$-Lipschitz continuous functions.

- Initial datum
  \[ x^0 \in C_s([-S^0, 0]; \mathbb{R}^d)^N \]
  with
  \[ S^0 := \frac{d_x(0)}{s - c}, \quad d_x(t) := \max_{i,j \in \{1, \ldots, N\}} |x_i(t) - x_j(t)| \]

- **Global** due to
  \[ |\dot{x}_i| \leq \sup_{r > 0} \psi(r)r = s < c \]
Theorem. In 1D, let $s < c$ and $\psi > 0$. Then

$$\lim_{t \to \infty} d_x(t) = 0$$

exponentially with explicit rate; but no conservation of mean.

Proof based on preservation of ordering

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_N(t)$$

so that $d_x = x_N - x_1$,

and on the monotonicity property

$$x_i < x_j \Rightarrow x_i < \tilde{x}_j^i, \quad \tilde{x}_i^j < x_j, \quad \tilde{x}_i^j < \tilde{x}_j^i$$

so that

$$\dot{x}_1 \geq 0, \quad \dot{x}_N \leq 0, \quad \frac{d}{dt} d_x(t) \leq 0.$$
Theorem. Let $s < c$. Then

$$\frac{d}{dt} d_x \leq \left( \frac{2s}{c - s} \bar{\psi} - \psi \right) d_x$$

with

$$\psi := \min_{r \in [0, d_x(0)]} \psi(r), \quad \bar{\psi} := \max_{r \in [0, d_x(0)]} \psi(r).$$

Consequently, exponential convergence to consensus whenever

$$\frac{2s}{c - s} < \frac{\psi}{\bar{\psi}}$$

In particular, $3s < c$ is necessary.

Proof based on triangle and Cauchy-Schwarz inequalities,

$$|\tilde{x}_j - x_j| \leq \frac{s}{c - s} d_x(t)$$
Which mean-field limit?
Collaboration with O. Tse (Eindhoven)
Remark. For the system with a fixed delay $\tau > 0$ the mean-field limit is given by the Fokker-Planck equation

$$\partial_t f_t + \nabla \cdot (F[f_{t-\tau}] f_t) = 0$$

for $f \in C([-\tau, T]; \mathcal{P}(\mathbb{R}^d))$, with

$$F[f_{t-\tau}](x) = \int_{\mathbb{R}^d} \psi(|x - y|)(x - y) \, df_{t-\tau}(y)$$

"Intuitive guess" (WRONG!) for our system:

$$\partial_t f_t + \nabla_x \cdot (G_t[f] f_t) = 0,$$

with

$$G_t[f](x) = \int_{\mathbb{R}^d} \psi(|x - y|)(x - y) f(t - c^{-1}|x - y|, y) \, dy$$
Mean-field limit

- Description in terms of $\varrho \in \mathcal{P}(\Omega_5)$ with

$$\Omega_5 := \{s - \text{Lipschitz continuous functions on } (\mathbb{R}, T]\}$$

- Denote $K(z) := \psi(|z|)z$.

- Study the object $x \in \Omega_5$ such that

$$\dot{x}(t) = \int_{\Omega_5} K(\Gamma_{t,x}(\gamma) - x(t)) \, d\varrho(\gamma)$$

with $\Gamma_{t,x} : \Omega_5 \mapsto \mathbb{R}^d$ defined as the unique solution $y \in \mathbb{R}^d$ of

$$y = \gamma(t - c^{-1}|x - y|)$$

- Using the Banach contraction theorem, construct $\varrho = \text{law}(x)$. 

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Consensus with Finite Speed Propagation
Definition of $\Gamma_{t,x}$
Goal: Pass from $\rho \in \mathcal{P}(\Omega_s)$ to $f \in C([−\infty, T]; \mathcal{P}(\mathbb{R}^d))$.

Define $f_t$ as the time-slice $f_t := T_t \# \rho \in \mathcal{P}(\mathbb{R}^d)$ with $T_t : \gamma \mapsto \gamma(t)$.

Question: Given a solution $\rho = \text{law}(x)$, can we write a closed equation for $f_t := T_t \# \rho$?

Answer: Yes, if we were able to express

$$G_t[\rho](x) := \int_{\Omega_s} K(\Gamma_{t,x}[\gamma] - x) \, d\rho(\gamma)$$

in terms of $f_t$. 
Fokker-Planck equation?

- Apply a co-ordinate transform $t \mapsto \tilde{t}_x$ which turns $\Gamma_{t,x}$ into $T_{\tilde{t}_x}$, i.e.,

$$\Gamma_{t,x}[\gamma] = T_{\tilde{t}_x}[\gamma] = \gamma(\tilde{t}_x)$$

... but (of course) $\tilde{t}_x$ depends on $\gamma$, i.e., $\gamma(\tilde{t}_x[\gamma])$

- One may integrate in time, which gives (drop $x$ for simplicity)

$$\int_{\Omega_s} \left( \int_{-\infty}^{\infty} \psi(t, \Gamma_t(\gamma)) \, dt \right) \, d\varrho(\gamma)$$

$$= \int_{\Omega_s} \int_{-\infty}^{\infty} \psi(\tilde{t} + c^{-1}|\gamma(\tilde{t})|, \gamma(\tilde{t})) \left( \frac{dt}{d\tilde{t}}[\gamma] \right) \, d\tilde{t} \, d\varrho(\gamma),$$

with a test function $\psi = \psi(t, y)$ and

$$\frac{dt}{d\tilde{t}}[\gamma] = 1 + c^{-1} \frac{\gamma(\tilde{t}) \cdot \dot{\gamma}(\tilde{t})}{|\gamma(\tilde{t})|}$$

- So the answer is: No (classical) Fokker-Planck equation.
Thank you for your attention!