Structured Realization Based on Time-Domain Data

8ECM

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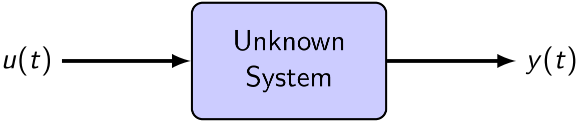
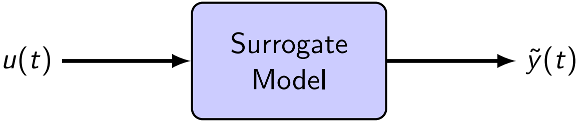
Data-Driven Reduced-Order Modeling



Task: Based on input/output measurements, construct a low-dimensional surrogate model



such that $\|y - \tilde{y}\|$ is small for all admissible inputs u.

Considered Class of Structures for the Surrogate Model

- we consider SISO, LTI systems with transfer functions of the form

$$H(s) = \boldsymbol{c}^\top \left(\sum_{k=1}^{K} \mathfrak{h}_k(s) A_k \right)^{-1} \boldsymbol{b}$$

- examples:
  - first-order systems

$$H(s) = \boldsymbol{c}^\top \left(sA_1 + A_2 \right)^{-1} \boldsymbol{b}$$

  - second-order systems

$$H(s) = \boldsymbol{c}^\top \left(s^2 A_1 + sA_2 + A_3 \right)^{-1} \boldsymbol{b}$$

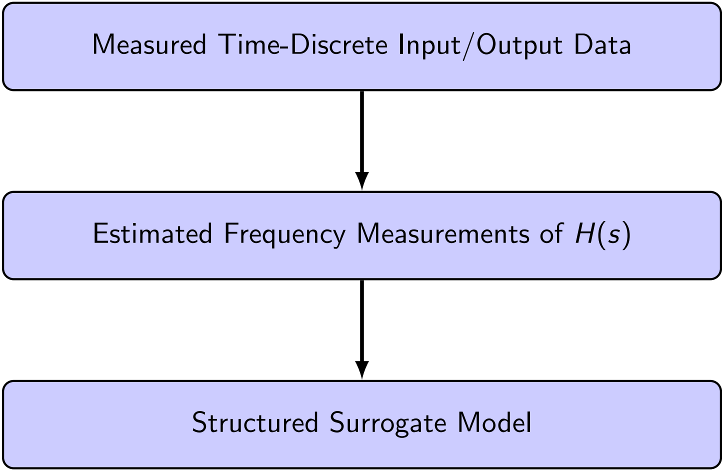
  - fractional-order systems

$$H(s) = \boldsymbol{c}^\top \left(\sum_{k=1}^{K} s^{\alpha_k} A_k \right)^{-1} \boldsymbol{b}$$

  - systems with time delay

$$H(s) = \boldsymbol{c}^\top \left(sA_1 + A_2 + e^{-\tau s} A_3 \right)^{-1} \boldsymbol{b}$$

Road Map for this Talk



Realization by the Loewner Framework

Realization based on Frequency-Domain Data

- LTI first-order systems [Mayo, Antoulas '07], [Lefteriu, Antoulas '10], [Beattie, Gugercin '12]
- parameter-dependent systems [Ionita, Antoulas '14]
- time-delay systems [Pontes Duff, Poussot-Vassal, Seren '15], [S., Unger '16]
- bilinear systems [Antoulas, Gosea, Ionita '16]
- quadratic-bilinear systems [Gosea, Antoulas '18]
- structured LTI systems [S., Unger, Beattie, Gugercin '18]
- switched systems [Gosea, Petreczky, Antoulas '18]
- LPV systems [Gosea, Petreczky, Antoulas '21]

Realization based on Time-Domain Data

- LTI first-order systems [Lefteriu, Ionita, Antoulas '10], [Peherstorfer, Gugercin, Willcox '17]
- bilinear systems [Karachalios, Gosea, Antoulas '20]

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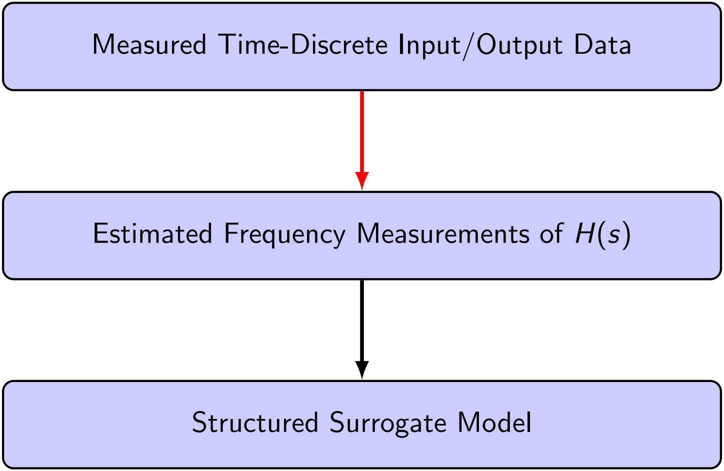

Transfer Function Estimation based on Given I/O Data (I)

- assume we are given time-discrete input/output data

$$u_j = u(j\delta_t), \quad y_j = y(j\delta_t) = \sum_{i=0}^{j} h_i u_{j-i} \quad \text{for } j = 0, \ldots, N-1$$

from a causal, BIBO stable, LTI system with zero initial value

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  from a causal, BIBO stable, LTI system with zero initial value
- using the discrete Fourier series

$$u_j = \frac{1}{N} \sum_{k=0}^{N-1} \hat{u}_k q_k^j = \frac{1}{N} \sum_{k \in \mathcal{I}} \hat{u}_k q_k^j \quad \text{with } q_k := \exp\left(\frac{2\pi i k}{N}\right)$$

  and $H_j(z) := \sum_{k=0}^{j} h_k z^{-k}$ we find

$$y_j = \sum_{i=0}^{j} h_i u_{j-i} = \frac{1}{N} \sum_{k \in \mathcal{I}} \hat{u}_k H_j(q_k) q_k^j \quad \text{for } j = 0, \dots, N-1$$

Transfer Function Estimation based on Given I/O Data (II)

$$y_j = \frac{1}{N} \sum_{k \in \mathcal{I}} \hat{u}_k H_j(q_k) q_k^j \quad \text{with } q_k := \exp\left(\frac{2\pi i k}{N}\right)$$

Theorem

Suppose the data u_j, y_j, $j = 0, \ldots, N-1$ stem from a causal, BIBO stable LTI system. Then, there holds

$$\lim_{j \to \infty} H_j(z) = H_{\mathcal{Z}}(z) \qquad \text{for all } z \in \mathbb{S} := \{z \in \mathbb{C} \mid |z| = 1\}.$$

- proposed method (cf. [1]): estimate $H_{\mathcal{Z}}(q_k)$ for $k \in \mathcal{I}$ by solving

$$\operatorname*{arg\,min}_{(\hat{H}_k)_{k \in \mathcal{I}}} \sum_{j=j_{\min}}^{N-1} \left| y_j - \frac{1}{N} \sum_{k \in \mathcal{I}} \hat{u}_k \hat{H}_k q_k^j \right|^2$$

[1] Peherstorfer, Gugercin, Willcox. Data-driven reduced model construction with time-domain Loewner models, *SIAM J. Sci. Comput.*, 39(5): A2152-A2178, 2017.

Choice of the Input Signal

- using the input signal

$$u(t) = \frac{1}{N}\sum_{k\in\mathcal{I}} \exp\left(\frac{2\pi ikt}{N\delta_t}\right) \quad \text{with } u_j := u(j\delta_t) = \frac{1}{N}\sum_{k\in\mathcal{I}} q_k^j$$

  yields the output

$$y_j := y(j\delta_t) = \int_0^{j\delta_t} h(j\delta_t - \sigma)u(\sigma)\,\mathrm{d}\sigma = \int_0^{j\delta_t} h(\sigma)u(j\delta_t - \sigma)\,\mathrm{d}\sigma$$

$$= \frac{1}{N}\sum_{k\in\mathcal{I}} q_k^j \int_0^{j\delta_t} h(\sigma)\exp\left(-\frac{2\pi ik\sigma}{N\delta_t}\right)\mathrm{d}\sigma \overset{j\gg 1}{\approx} \frac{1}{N}\sum_{k\in\mathcal{I}} q_k^j H\left(\frac{2\pi ik}{N\delta_t}\right)$$

- discrete input signal has the property $\hat{u}_k = 1$ if $k \in \mathcal{I}$ and $\hat{u}_k = 0$ else

Algorithm for the Transfer Function Estimation

Input: frequencies of interest $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\tilde{r}} \in i\mathbb{R}$; $j_{\min}$; δ_t; N; β
Output: derived frequencies $\lambda_1, \ldots, \lambda_r$ and corresponding transfer function estimates

1: Solve

$$\lambda_j = \underset{s \in \{0, \frac{2\pi i}{N\delta_t}, \ldots, \frac{2\pi i(N-1)}{N\delta_t}\}}{\arg\min} \left| s - \tilde{\lambda}_j \right|^2 \quad \text{for } j = 1, \ldots, \tilde{r}.$$

2: Remove redundant frequencies to obtain $\lambda_1, \ldots, \lambda_r$ with $r \leq \tilde{r}$.

3: Construct input signal as on last slide and obtain u_j, y_j, $j = 0, \ldots, N-1$.

4: Compute Fourier coefficients of the input signal via an FFT.

5: Solve the least squares problem

$$\underset{(\hat{H}_k)_{k \in \mathcal{I}}}{\arg\min} \sum_{j=j_{\min}}^{N-1} \left| y_j - \frac{1}{N} \sum_{k \in \mathcal{I}} \hat{u}_k \hat{H}_k q_k^j \right|^2.$$

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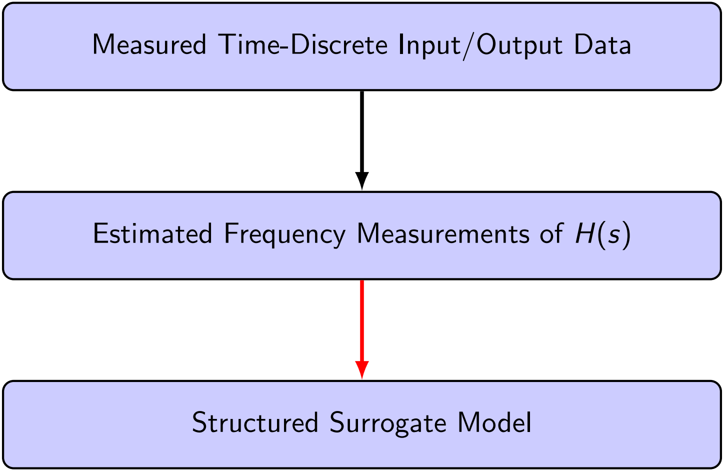


The Structured Realization Problem [2]

- assume we are given transfer function evaluations

$$H(\lambda_j) = \theta_j \quad \text{for } j = 1, \ldots, Kn$$

  and functions $\mathfrak{h}_1$, ..., $\mathfrak{h}_K$ defining the structure
- task: find $A_1, \ldots, A_K \in \mathbb{C}^{n,n}$ and $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}^n$ such that

$$\tilde{H}(s) := \boldsymbol{c}^\top \left(\sum_{k=1}^{K} \mathfrak{h}_k(s) A_k \right)^{-1} \boldsymbol{b}$$

  satisfies $H(\lambda_j) = \tilde{H}(\lambda_j)$ for $j = 1, \ldots, Kn$
- special case $K = 2$, $\mathfrak{h}_1(s) := s$, $\mathfrak{h}_2(s) := -1$ is solved in [3]

[2] S., Unger, Beattie, Gugercin. Data-driven structured realization, *Linear Algebra Appl.*, 537: 250–286, 2018.

[3] Mayo, Antoulas. A framework for the solution of the generalized realization problem, *Linear Algebra Appl.*, 425: 634–662, 2007.

Main Idea of the Realization Procedure from [2] (I)

- divide the given transfer function data into two sets

$$(\lambda_\ell,\, \theta_\ell = H(\lambda_\ell))_{\ell=1}^{q_\ell n}, \quad (\sigma_j,\, \zeta_j = H(\sigma_j))_{j=1}^{q_\mathrm{r} n} \quad \text{with } q_\ell + q_\mathrm{r} = K$$

- $\tilde{H}(\sigma_j)$ can be written as $\tilde{H}(\sigma_j) = \boldsymbol{c}^\top \underbrace{\left(\sum_{k=1}^{K} \mathfrak{h}_k(\sigma_j) A_k\right)^{-1} \boldsymbol{b}}_{=:\boldsymbol{p}_{\mathrm{r};j}}$

- $\tilde{H}(\sigma_j) = \zeta_j$ is satisfied iff there exists $\boldsymbol{p}_{\mathrm{r};j} \in \mathbb{C}^{n,1}$ s.t.

$$\zeta_j = \boldsymbol{c}^\top \boldsymbol{p}_{\mathrm{r};j} \quad \text{and} \quad \sum_{k=1}^{K} \mathfrak{h}_k(\sigma_j) A_k \boldsymbol{p}_{\mathrm{r};j} = \boldsymbol{b}$$

[2] S., Unger, Beattie, Gugercin. Data-driven structured realization, *Linear Algebra Appl.*, 537: 250-286, 2018.

Main Idea of the Realization Procedure from [2] (II)

- all interpolation conditions are met iff there exist P_{r}, P_{ℓ} s.t.

$$\mathbf{1}^\top Z = \boldsymbol{c}^\top P_{\mathrm{r}}, \qquad \sum_{k=1}^{K} A_k P_{\mathrm{r}} \mathfrak{h}_k(\Sigma) = \boldsymbol{b}\mathbf{1}^\top$$

$$\Theta \mathbf{1} = P_{\ell}^\top \boldsymbol{b}, \qquad \sum_{k=1}^{K} \mathfrak{h}_k(\Lambda) P_{\ell}^\top A_k = \mathbf{1}\boldsymbol{c}^\top$$

- by fixing P_{r} and P_{ℓ} this becomes a linear system for $A_1, \ldots, A_K, \boldsymbol{b}, \boldsymbol{c}$
- for further details (e.g. MIMO case, real-valued realizations), see [2]

[2] S., Unger, Beattie, Gugercin. Data-driven structured realization, *Linear Algebra Appl.*, 537: 250-286, 2018.

Estimation of Realization Parameters from Data

- system structure may depend on a parameter, e.g., for delay systems

$$H(s,\tau) = \boldsymbol{c}^\top \left(sA_1 + A_2 + e^{-\tau s} A_3\right)^{-1} \boldsymbol{b}$$

- for each fixed $\hat{\tau}$ value we may construct a realization $\tilde{H}(s,\hat{\tau})$
- idea: based on additional transfer function measurements

$$(\gamma_j,\, \eta_j = H(\gamma_j,\tau))_{j=1}^q$$

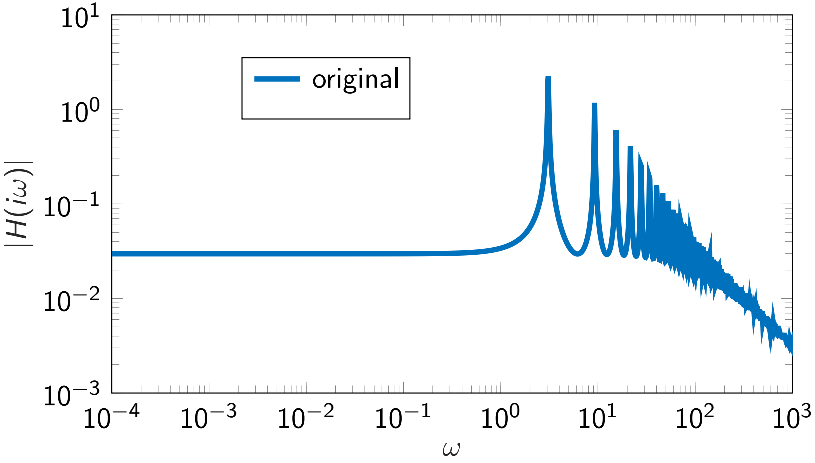
we estimate τ by solving the nonlinear least squares problem

$$\min_{\hat{\tau}\in\mathbb{T}} \sum_{j=1}^q \left|\eta_j - \tilde{H}(\gamma_j,\hat{\tau})\right|^2$$

Numerical Example with a Time Delay

- considered delay system has state dimension 12 and transfer function

$$H(s) = \boldsymbol{c}^\top \left(sA_1 + A_2 + e^{-\tau s} A_3\right)^{-1} \boldsymbol{b} \quad \text{with } \tau = 1$$



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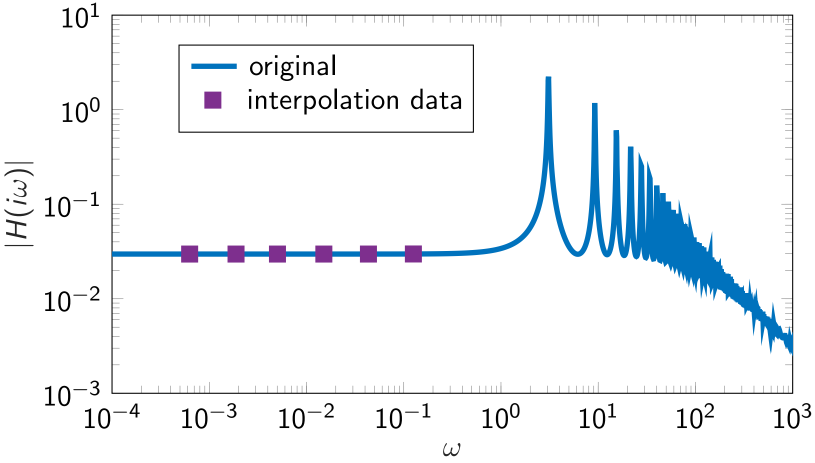

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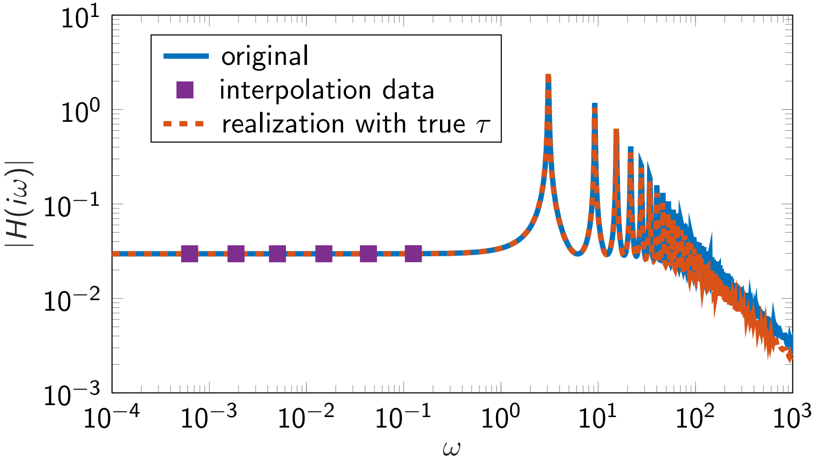

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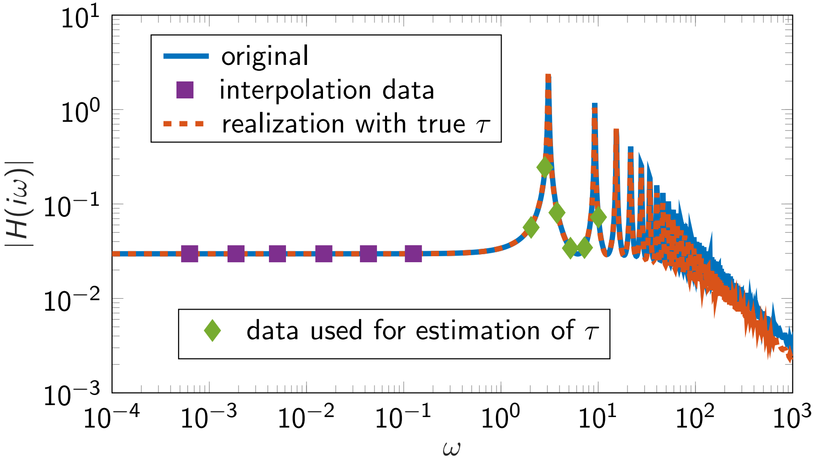

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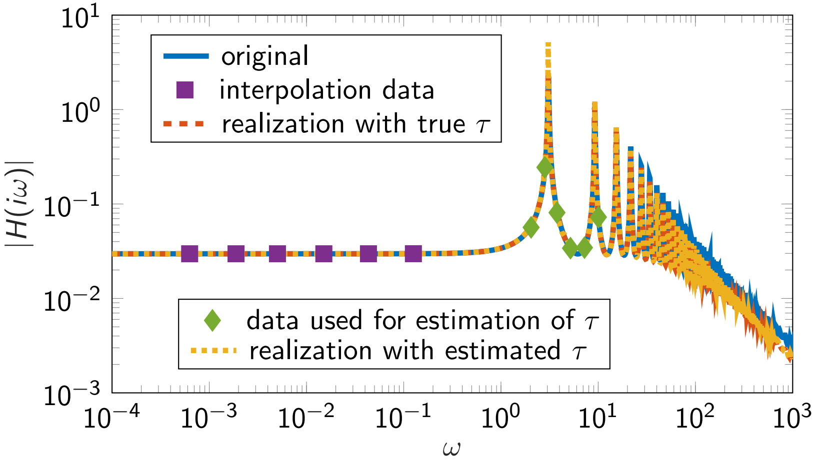
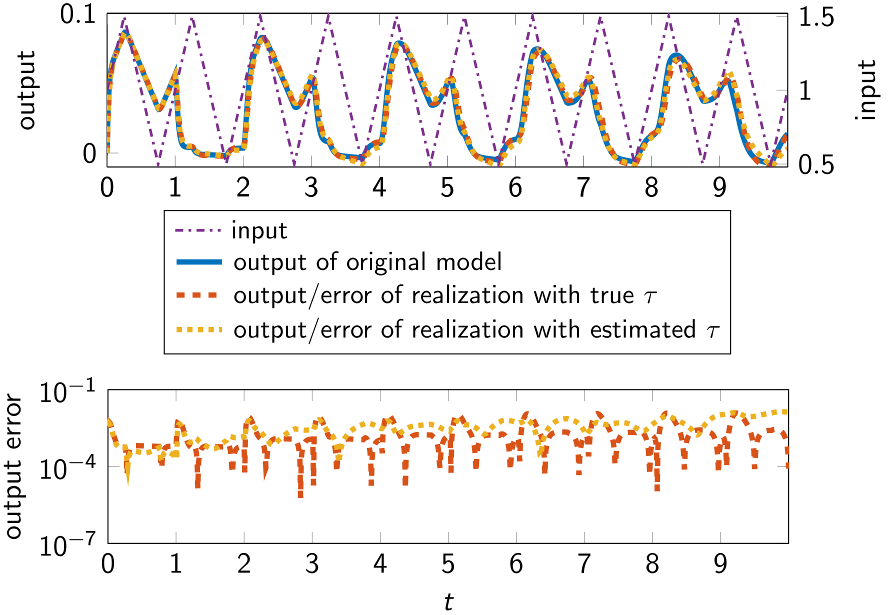
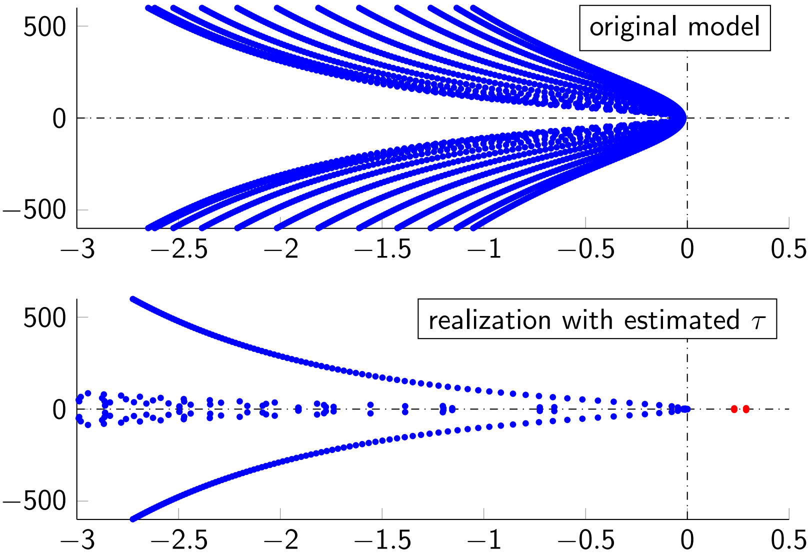


Comparison in the Time Domain



Comparison of the Transfer Function Poles



Conclusion

Summary

- proposed method to obtain structured LTI systems from time-domain data [4]
- first numerical results for a time-delay test case look promising

Outlook

- enforcement of stability
- application to other structures and noisy data

[4] Fosong, S., Unger. From time-domain data to low-dimensional structured models. ArXiv preprint 1902.05112, 2019.