# Čech cohomology and the region of influence of non-saddle sets

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# General framework

In this talk we consider flows $\varphi : M \times \mathbb{R} \longrightarrow M$ where $M$ is a locally compact ANR. Recall that an ANR $M$ is a metric space that satisfies the following:

- Whenever there exists an embedding $f : M \to Y$ of $M$ into a metric space $Y$ such that $f(M)$ is closed in $Y$, there exists a neighborhood $U$ of $f(M)$ such that $f(M)$ is a retract of $U$.

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# Isolated non-saddle sets

We are interested in the global structure of flows that possess a special type of invariant sets the so-called *isolated non-saddle sets*. An invariant compactum $K$ is said to be

- *saddle* whenever there exist points arbitrarily close to $K$ whose trajectories get far from $K$ in both the future and the past. Otherwise $K$ is said to be *non-saddle*.
- an *isolated invariant set* if it possesses a so-called *isolating neighborhood*, that is, a compact neighborhood $N$ such that $K$ is the maximal invariant set in $N$.

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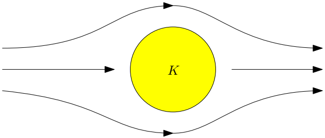
# Isolated non-saddle set

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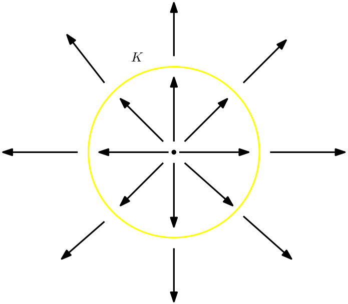


Isolated non-saddle set

# Isolated non-saddle set

- The theory of non-saddle sets was first studied by Bhatia and Ura, although, according to Ura, it was introduced by Seibert in an oral communication.
- Stable attractors, negatively stable repellers and some unstable attractors are examples of non-saddle sets.
- It was proven by Giraldo, Morón, Ruiz del Portal and Sanjurjo that, while every finite dimensional compactum can be realized as an isolated saddle set for a flow in some euclidean space, isolated non-saddle sets have the Borsuk's homotopy type of finite polyhedra.

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To deal with isolated invariant sets we make use of a special type of isolating neighborhoods, the so-called isolating blocks. An *isolating block* $N$ is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and exit sets, satisfying

1. $\partial N = N^i \cup N^o$,
2. The trajectory of every point $x \in N^i \setminus N^o$ enters in the interior of $N$ inmediately in positive time and leaves $N$ inmediately in negative time.
3. The trajectory of every point $x \in N^o \setminus N^i$ enters in the interior of $N$ inmediately in negative time and leaves $N$ inmediately in positive time.
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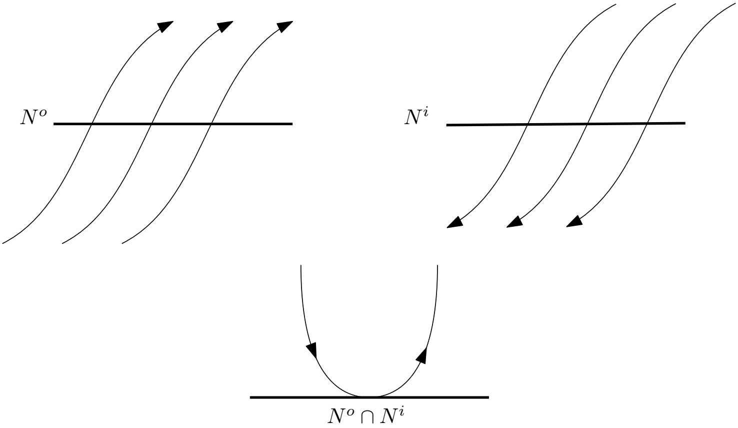
$N^o$
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# Local structure of non-saddle sets

Given an isolated non-saddle set $K$ we can always find an isolating block $N$ of the form $N^+ \cup N^-$ where

$$N^+ = \{x \in N \mid x[0,+\infty) \subset N\}, \quad N^- = \{x \in N \mid x(-\infty,0] \subset N\}.$$

# Local structure of non-saddle sets

The following properties are consequences of the previous remark

1. Each component of $N \setminus K$ is either attracted or repelled by $K$.
2. The flow provides a deformation retraction from $N \setminus K$ onto $\partial N$.
3. $i : K \hookrightarrow N$ is a Borsuk's homotopy equivalence.

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# The region of influence of an isolated non-saddle set

The *region of influence* of an isolated non-saddle set is defined as the set

$$\mathcal{I}(K) = W^s(K) \cup W^u(K)$$

and is an open invariant subset of the phase space.

# The region of influence of an isolated non-saddle set

The region of influence $\mathcal{I}(K)$ is composed of three different kinds of points.

1. *Purely attracted points*, that is, points $x \in \mathcal{I}(K)$ with $\omega(x) \subset K$ and $\omega^*(x) \not\subset K$.
2. *Purely repelled points*, that is, points $x \in \mathcal{I}(K)$ with $\omega^*(x) \subset K$ and $\omega(x) \not\subset K$.
3. *Homoclinic points*, that is, points $x \in \mathcal{I}(K)$ with $\omega^*(x) \subset K$ and $\omega(x) \subset K$.

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# The region of influence of an isolated non-saddle set

$\mathcal{I}(K) \setminus K$ has a finite number of components and each component is of one of the following types:

1. Every point is purely attracted (*purely attracted component*).
2. Every point is purely repelled (*purely repelled component*).
3. Every point is homoclinic (*homoclinic component*).
4. It contains points of the three types (*dissonant component*).

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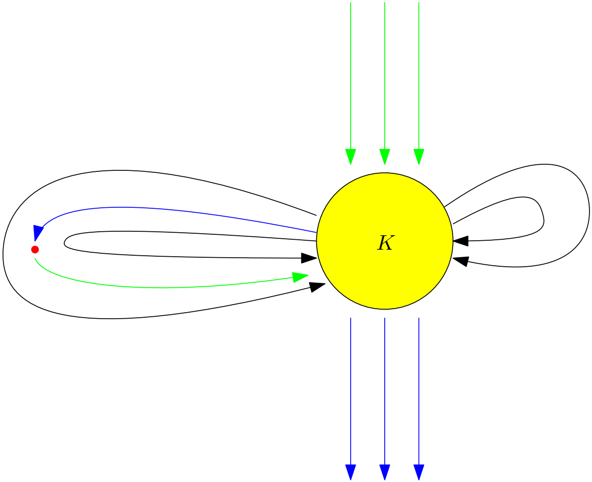

# The region of influence of an isolated non-saddle set

**Theorem**

*Let $M$ be a connected locally compact ANR and $K$ a connected isolated non-saddle set of a flow on $M$. Suppose that $H^1(M; G) = 0$ or, more generally, that the homomorphism $i^{*_1} : \check{H}^1(M; G) \to \check{H}^1(K; G)$ induced in 1-dimensional Čech cohomology by the inclusion $i : K \hookrightarrow M$ is a monomorphism. Then $\mathcal{I}(K) \setminus K$ does have neither homoclinic nor dissonant components. Moreover, if $U$ is a component of $M \setminus K$, then the flow restricted to $U$ is either locally attracted by $K$ (i.e. all points lying in $U$ near $K$ are attracted by $K$) or locally repelled by $K$. Furthermore, if $N$ is an isolating block of $K$ of the form $N = N^+ \cup N^-$ then each component of $M \setminus K$ contains exactly one component of $\partial N$.*

# The complexity of the region of influence

## Definition

Let $M$ be a locally compact ANR and suppose that $K$ is an isolated non-saddle set of a flow on $M$ and $N$ and isolating block of the form $N^+ \cup N^-$. We define the *complexity* of $\mathcal{I}(K)$ as the difference $k - m$ where $k$ denotes the number of components of $N \setminus K$ and $m$ denotes the number of components of $\mathcal{I}(K) \setminus K$.

# The complexity of the region of influence

## Theorem

*Let $K$ be an isolated non-saddle continuum of a flow defined on a connected locally compact ANR $M$ and $i^{*_k}: \check{H}^k(M;G) \to \check{H}^k(K;G)$ the homomorphism induced in $k$-dimensional Čech cohomology by the inclusion $i: K \hookrightarrow M$. Suppose that the complexity of the region of influence of $K$ is $\mathfrak{c}$. Then there exist*

$$\alpha_1, \ldots, \alpha_{\mathfrak{c}} \in \check{H}^1(M;G)$$

*which are independent non-torsion elements satisfying that $i^{*_1}(\alpha_j) = 0$ for every $j = 1, \ldots, \mathfrak{c}$. Moreover, if $M$ is a closed, connected and $G$-orientable $n$-manifold, then there exist*

$$\beta_1, \ldots, \beta_{\mathfrak{c}} \in \check{H}^{n-1}(M;G) \quad \text{and} \quad \gamma_1, \ldots, \gamma_{\mathfrak{c}} \in \check{H}^{n-1}(K;G)$$

*which are independent non-torsion elements such that $i^{*_{n-1}}(\beta_j) = \gamma_j$ for each $j = 1, \ldots, \mathfrak{c}$.*

# The complexity of the region of influence on the $n$-torus

## Proposition

*Suppose $K$ is an isolated non-saddle continuum in the n-dimensional torus $T^n$. Then the complexity of $\mathcal{I}(K)$ is at most 1.*

**Proposition**

*Let $K$ be a connected isolated non-saddle set of a flow defined on a closed, orientable surface $M$ of genus $g$. Then the complexity of $\mathcal{I}(K)$ is at most $g$.*

Thank you very much for your attention!