On some fractional problems with Dirichlet-Neumann boundary conditions

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- Brezis-Nirenberg fractional problem with mixed Dirichlet-Neumann (D-N) boundary conditions
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- Definition of the Fractional Laplacian (through the spectral decomposition) and extended problem to one more variable
- Sobolev and Trace Inequalities
- Attainability of the Sobolev constant
- Brezis-Nirenberg fractional problem with mixed Dirichlet-Neumann (D-N) boundary conditions
- Fractional elliptic problems involving an inverse fractional operator
Fractional Laplacian with D-N boundary data

Powers of Laplacian operator \((-\Delta)\):

Let \((\lambda_n, \varphi_n)\) be the eigenvalues and eigenfunctions of \((-\Delta)\) in \(\Omega\) with zero mixed D-N boundary data. Then \((\lambda_n^s, \varphi_n)\) are the eigenvalues and eigenfunctions of \((-\Delta)^s\), also with zero D-N boundary conditions.
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The fractional Laplacian \((-\Delta)^s\) is well defined in the space of functions that vanish on \(\Sigma_D\),

\[
H^s_{\Sigma_D}(\Omega) = \left\{ u = \sum_{n \geq 1} a_n \varphi_n \in L^2(\Omega) : \|u\|_{H^s_{\Sigma_D}(\Omega)}^2 = \sum_{n \geq 1} a_n^2 \lambda_n^s < \infty \right\}.
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\]

As a consequence,

\[
(-\Delta)^s u = \sum_{n \geq 1} \lambda_n^s a_n \varphi_n.
\]

Note that then \(\|u\|_{H_{\Sigma_D}^s(\Omega)} = \|(-\Delta)^{s/2} u\|_{L^2(\Omega)}.\)
Fractional Laplacian with D-N boundary data

Following [LM]

- $H^s_0(\Omega) = H^s(\Omega)$ for $0 < s \leq \frac{1}{2}$.
- $H^s_0(\Omega) \subsetneq H^s(\Omega)$ for $\frac{1}{2} < s < 1$.

Fractional Laplacian with D-N boundary data

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\[ H^s_0(\Omega) = H^s(\Omega) \text{ for } 0 < s \leq \frac{1}{2}. \]

\[ H^s_0(\Omega) \subsetneq H^s(\Omega) \text{ for } \frac{1}{2} < s < 1. \]


As a consequence

\[ H^s_{\Sigma_D}(\Omega) = H^s(\Omega) \text{ for } 0 < s \leq \frac{1}{2}. \]

\[ H^s_{\Sigma_D}(\Omega) \subsetneq H^s(\Omega) \text{ for } \frac{1}{2} < s < 1. \]
For the general problem

\[
(P) \quad \begin{cases}
(-\Delta)^s u = f(x, u) & \text{in } \Omega, \\
B(u) = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where we take mixed Dirichlet-Neumann boundary conditions,

\[B(u) = \chi_{\Sigma_D} u + \chi_{\Sigma_N} \frac{\partial u}{\partial \nu}.\]
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B(u) = \chi_{\Sigma_D} u + \chi_{\Sigma_N} \frac{\partial u}{\partial \nu}.
\]

- \(\Sigma_D\) and \(\Sigma_N\) are smooth \((N-1)\)-dimensional submanifolds of \(\partial\Omega\).
- \(\Sigma_D\) is a closed manifold of positive \((N-1)\)-dimensional Hausdorff measure,

\[
\mathcal{H}^{N-1}(\Sigma_D) = \alpha \in (0, \mathcal{H}^{N-1}(\partial\Omega)).
\]

- \(\Sigma_D\) and \(\Sigma_N\) verify \(\Sigma_D \cap \Sigma_N = \emptyset\), \(\Sigma_D \cup \Sigma_N = \partial\Omega\), \(\Sigma_D \cap \overline{\Sigma_N} = \Gamma\), where \(\Gamma\) is a smooth \((N-2)\)-dimensional submanifold of \(\partial\Omega\).
Fractional Laplacian with D-N boundary data

\[ (P_{\lambda}) \begin{cases} \begin{align*}
(-\Delta)^s u &= \lambda u + u \frac{N+2s}{N-2s}, \quad u > 0 \quad \text{in } \Omega, \\
B(u) &= 0 \quad \text{on } \partial \Omega,
\end{align*} \end{cases} \]

where \( \lambda > 0 \), and \( \Omega \subset \mathbb{R}^N \), with \( N > 2s \), \( \frac{1}{2} < s < 1 \).

Fractional Laplacian with D-N boundary data

\[(P_\lambda) \begin{cases} (-\Delta)^s u = \lambda u + u \frac{N+2s}{N-2s}, & u > 0 \quad \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial \Omega, \end{cases}\]

where \(\lambda > 0\), and \(\Omega \subset \mathbb{R}^N\), with \(N > 2s, \frac{1}{2} < s < 1\).


Sense of weak/energy solution

\[\int_\Omega (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx = \int_\Omega f_\lambda(u) \varphi \, dx, \quad \forall \varphi \in H^s_{\Sigma_D}(\Omega).\]

We also have an associated energy functional \((2^* = \frac{2N}{N-2s})\)

\[I(u) = \frac{1}{2} \int_\Omega \left| (-\Delta)^{s/2} u \right|^2 \, dx - \frac{\lambda}{2} \int_\Omega u^2 \, dx - \frac{1}{2^*} \int_\Omega u^{2^*} \, dx\]

which is well defined in \(H^s_{\Sigma_D}(\Omega)\). Clearly, the critical points of \(I\) correspond to solutions to \((P_\lambda)\).
Extended problems to one more variable

Consider the cylinder $C_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}^{N+1}_+$. Given $u \in H^s_{\Sigma_D}(\Omega)$, we define its $s$-harmonic extension $w = E_s(u)$ to the cylinder $C_\Omega$ as the solution to the problem

$$
\begin{cases}
-\text{div}(y^{1-2s}\nabla w) = 0 & \text{in } C_\Omega, \\
B^*(w) = 0 & \text{on } \partial_L C_\Omega = \partial \Omega \times [0,\infty), \\
w = u & \text{on } \Omega \times \{y = 0\}.
\end{cases}
$$

where

$$B^*(w) = w\chi_{\Sigma_D^*} + \frac{\partial w}{\partial \nu}\chi_{\Sigma_N^*},$$

with $\Sigma_D^* = \Sigma_D \times [0,\infty)$ and $\Sigma_N^* = \Sigma_N \times [0,\infty)$. 
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with $\Sigma_D^* = \Sigma_D \times [0, \infty)$ and $\Sigma_N^* = \Sigma_N \times [0, \infty)$.

The extension function belongs to the space $X^{s}_{\Sigma_D^*}(C_\Omega)$ defined as the completion of $\{z \in C^\infty(C_\Omega) : z = 0 \text{ on } \Sigma_D^*\}$ with respect to the norm

$$
\|z\|_{X^{s}_{\Sigma_D^*}(C_\Omega)} = \left( \kappa_s \int_{C_\Omega} y^{1-2s}|\nabla z|^2 \, dx \, dy \right)^{1/2}
$$

where $\kappa_s$ is a normalization constant.
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Note that the extension operator is an isometry

$$
\|E_s(\psi)\|_{X^s_{\Sigma_D^*}(C_{\Omega})} = \|\psi\|_{H^s_{\Sigma_D}(\Omega)}, \quad \forall \psi \in H^s_{\Sigma_D}(\Omega).
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Note that the extension operator is an isometry

$$\|E_s(\psi)\|_{X^{s}_{\Sigma_D^*}(C_\Omega)} = \|\psi\|_{H^s_{\Sigma_D}(\Omega)}, \quad \forall \psi \in H^s_{\Sigma_D}(\Omega).$$

Moreover, for any $\varphi \in X^{s}_{\Sigma_D^*}(C_\Omega)$, we have the following trace inequality

$$\|\varphi\|_{X^{s}_{\Sigma_D^*}(C_\Omega)} \geq \|\varphi(\cdot, 0)\|_{H^s_{\Sigma_D}(\Omega)}.$$
Extended problems to one more variable

The relevance of the extension function $w$ is that it is related to the fractional Laplacian of the original function $u$ through the formula

$$-\kappa_s \lim_{y \searrow 0} y^{1-2s} \frac{\partial w}{\partial y}(x, y) = (-\Delta)^s u(x),$$
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Extended problems to one more variable

Denoting

\[ L_s w := -\text{div}(y^{1-2s} \nabla w) , \quad \frac{\partial w}{\partial \nu^s} := -\kappa_s \lim_{y \searrow 0} y^{1-2s} \frac{\partial w}{\partial y} \]

we can reformulate \((P_\lambda)\) with the new variable as

\[
(P^*_\lambda) \quad \begin{cases} 
L_s w = 0 & \text{in } C_\Omega, \\
B^*(w) = 0 & \text{on } \partial L C_\Omega, \\
\frac{\partial w}{\partial \nu^s} = \lambda w + w^{\frac{N+2s}{N-2s}} & \text{in } \Omega \times \{y = 0\}. 
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\end{cases}
\]

We say as before that \(w \in X^{s}_{\Sigma^*D}(C_\Omega)\) is an energy solution if

\[
\kappa_s \int_{C_\Omega} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle \, dx \, dy = \int_{\Omega} \left( \lambda w + w \frac{N+2s}{N-2s} \right) \varphi \, dx, \quad \forall \varphi \in X^s_{\Sigma D}(C_\Omega).
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\]

Energy functional

\[
J(w) = \frac{\kappa_s}{2} \int_{C_\Omega} y^{1-2s} |\nabla w|^2 \, dx \, dy - \frac{\lambda}{2} \int_{\Omega} w^2 \, dx - \frac{1}{2^*_s} \int_{\Omega} w^{2^*_s} \, dx.
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Extended problems to one more variable

Denoting

\[ L_sw := -\text{div}(y^{1-2s}\nabla w), \quad \frac{\partial w}{\partial \nu^s} := -\kappa_s \lim_{y \searrow 0} y^{1-2s} \frac{\partial w}{\partial y} \]

we can reformulate \((P_\lambda)\) with the new variable as

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\end{cases}
\]

We say as before that \(w \in X_{\Sigma_D^*}^s (C_\Omega)\) is an energy solution if

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\kappa_s \int_{C_\Omega} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle \, dx \, dy = \int_{\Omega} \left( \lambda w + w^{\frac{N+2s}{N-2s}} \right) \varphi \, dx, \quad \forall \, \varphi \in X_{\Sigma_D}^s (C_\Omega).
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J(w) = \frac{\kappa_s}{2} \int_{C_\Omega} y^{1-2s} |\nabla w|^2 \, dx \, dy - \frac{\lambda}{2} \int_{\Omega} w^2 \, dx - \frac{1}{2^{*}} \int_{\Omega} w^{2^{*}} \, dx.
\]

Note that critical points of \(J\) in \(X_{\Sigma_D^*}^s (C_\Omega)\) correspond to critical points of \(I\) in \(H_{\Sigma_D^*}^s (\Omega)\).
Sobolev and Trace inequalities (Mixed D-N)

Since we have a Dirichlet condition on $\Sigma_D$ with $0 < \mathcal{H}^{N-1}(\Sigma_D) < \mathcal{H}^{N-1}(\partial \Omega)$, then

$$0 < C := \inf_{\begin{array}{c} \|u\|_{H^s_{\Sigma_D}(\Omega)} \\
\|u\|_{L^{2^*}_s(\Omega)} 
\end{array}} \frac{\|u\|_{H^s_{\Sigma_D}(\Omega)}}{\|u\|_{L^{2^*}_s(\Omega)}}.$$
Sobolev and Trace inequalities (Mixed D-N)

Since we have a Dirichlet condition on $\Sigma_D$ with $0 < \mathcal{H}^{N-1}(\Sigma_D) < \mathcal{H}^{N-1}(\partial\Omega)$, then

$$0 < C := \inf_{u \in H^s_{\Sigma_D}(\Omega), u \neq 0} \frac{\|u\|_{H^s_{\Sigma_D}(\Omega)}}{\|u\|_{L^{2^*_s}(\Omega)}}.$$  

Hence, in terms of the extension function,

$$\left(\int_{\Omega} \varphi^{\frac{2N}{N-2s}}(x, 0) \, dx\right)^{\frac{N-2s}{2N}} \leq C\|\varphi(\cdot, 0)\|_{H^s_{\Sigma_D}(\Omega)} = C\|E_s[\varphi(\cdot, 0)]\|_{X^s_{\Sigma_D}(\mathcal{C}_\Omega)}.$$
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Since we have a Dirichlet condition on $\Sigma_D$ with $0 < \mathcal{H}^{N-1}(\Sigma_D) < \mathcal{H}^{N-1}(\partial\Omega)$, then

$$0 < C := \inf_{u \in H^s_{\Sigma_D}(\Omega), u \not\equiv 0} \frac{\|u\|_{H^s_{\Sigma_D}(\Omega)}}{\|u\|_{L^{2^*}(\Omega)}}.$$ 

Hence, in terms of the extension function,

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As a consequence, we obtain the following **Mixed Trace inequality**, 

$$\left(\int_{\Omega} \varphi \frac{2N}{N-2s} (x, 0) dx \right)^{1-\frac{2s}{N}} \leq C \int_{C_{\Omega}} y^{1-2s} |\nabla \varphi|^2 dxdy.$$ 

for any $\varphi \in X^s_{\Sigma_D}(C_{\Omega})$, where $C$ is a positive constant.
Sobolev constant relative to $\Sigma_D$

We define the Sobolev constant "relative to $\Sigma_D$" as follows,

$$S(\Sigma_D) = \inf_{\substack{u \in H^s_{\Sigma_D}(\Omega) \setminus \{0\}}} \frac{\|u\|^2_{H^s_{\Sigma_D}(\Omega)}}{\|u\|^2_{L^{2s}(\Omega)}} = \inf_{\substack{w \in X^s_{\Sigma_D}(C\Omega) \setminus \{0\}}} \frac{\|w\|^2_{X^s_{\Sigma_D}(C\Omega)}}{\|w(\cdot,0)\|^2_{L^{2s}(\Omega)}}.$$
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**Theorem 1.** $S(\Sigma_D) \leq 2^{-\frac{2s}{N}} \kappa_s S(s, N)$, and even more, if $S(\Sigma_D) < 2^{-\frac{2s}{N}} \kappa_s S(s, N)$

$\Rightarrow S(\Sigma_D)$ is attained.
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**Theorem 1.** $S(\Sigma_D) \leq 2^{-\frac{2s}{N}} \kappa_s S(s, N)$, and even more, if $S(\Sigma_D) < 2^{-\frac{2s}{N}} \kappa_s S(s, N)$ \implies $S(\Sigma_D)$ is attained.

The key of the proof relies on concentration-compactness arguments by Lions [L]. See [ACP] for similar arguments adapted to mixed problems with $s = 1$.

Sobolev constant relative to $\Sigma_D$

Following [CP, Lemma 4.1] we have the next result.

**Lemma 1.** Under certain geometrical assumptions on the distribution of $\Sigma_D$, $\Sigma_N$ on $\partial \Omega$, $\lambda_1^s(\alpha) \to 0$, as $\alpha = \mathcal{H}^{N-1}(\Sigma_D) \to 0$.

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**Lemma 1.** Under certain geometrical assumptions on the distribution of $\Sigma_D$, $\Sigma_N$ on $\partial \Omega$, $\lambda_1^s(\alpha) \to 0$, as $\alpha = H^{N-1}(\Sigma_D) \to 0$.


**Lemma 2.** $S(\Sigma_D) \leq C\lambda_1^s(\alpha)$. 
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**Theorem 2.** Under some geometrical assumptions, the Sobolev constant $S(\Sigma_D)$ is attained.
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**Theorem 2.** Under some geometrical assumptions, the Sobolev constant $S(\Sigma_D)$ is attained.

The proof follows by using Theorem 1 and Lemmas 1-2 jointly because $S(\Sigma_D)$ is as small as we want provided $\alpha \to 0$, proving that $S(\Sigma_D) < 2^{-\frac{2s}{N}} \kappa_s S(s, N)$. 
Main Results

Remember the main problem

\[
(P_{\lambda}) \begin{cases} 
(\Delta)^s u = \lambda u + u^{\frac{N+2s}{N-2s}}, & u > 0 \quad \text{in } \Omega, \\
B(u) = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where \( \lambda > 0 \), and \( \Omega \subset \mathbb{R}^N \), with \( N > 2s, \frac{1}{2} < s < 1 \).
Main Results

**Theorem 3.** Assume that $\frac{1}{2} < s < 1$ and $N \geq 4s$. Then problem $(P_\lambda)$:

1. has no solution for $\lambda \geq \lambda_1^s$,
2. has solution for each $0 < \lambda < \lambda_1^s$,
3. under the some geometrical assumptions, has solution for $\lambda = 0$ and $\mathcal{H}^{N-1}(\Sigma_D)$ sufficiently small.
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3 has been already proved before because of the attainability of the Sobolev constant $S(\Sigma_D)$. 
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See [BN] for points 1, 2, with $s = 1$ and Dirichlet boundary data,

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1 can be easily proved by using the first eigenfunction as a test function.

3 has been already proved before because of the attainability of the Sobolev constant $S'(\Sigma_D)$.

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**Theorem 3.** Assume that $\frac{1}{2} < s < 1$ and $N \geq 4s$. Then problem $(P_\lambda)$:

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Variational approach: minimizers

To prove point 2 in Theorem 3, i.e., the existence of solution to $(P_\lambda)$, for $0 < \lambda < \lambda_1^s$, we consider the following quotient

$$Q_\lambda(w) = \frac{\|w\|_{X^{s}_{\Sigma D}(C_{\Omega})}^2 - \lambda\|u\|_{L^2(\Omega)}^2}{\|u\|_{L^{2^*_s}(\Omega)}^2},$$

where $w = E_s[u]$, and we define

$$S_\lambda(\Omega) = \inf_{\substack{w \in X^{s}_{\Sigma D}(C_{\Omega}) \backslash \{0\}}} \{Q_\lambda(w)\},$$

in order to find a minimizer.
Fractional elliptic problems, inverse fractional operator

\[
(P_{\alpha,\beta}) \begin{cases} 
(-\Delta)^{\alpha-\beta} u = \lambda (-\Delta)^{-\beta} u + |u|^{2^*_{\mu} - 2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

We prove existence or nonexistence of positive solutions depending on the parameter \( \lambda > 0 \), up to the critical value of the exponent \( p \), i.e., for \( 1 < p \leq 2^*_{\mu} - 1 \) where \( \mu := \alpha - \beta \) and \( 2^*_{\mu} = \frac{2N}{N - 2\mu} \) is the critical exponent of the Sobolev embedding.
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**Theorem.** For every \( \gamma \in (0, \lambda_1^{\alpha}) \), there exists a positive solution for the problem \((P_{\alpha,\beta})\) provided that \( N > 4\alpha - 2\beta \).
Thank you for the attention!