The epsilon constant conjecture for higher dimensional unramified twists of $\mathbb{Z}_p^r(1)$

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Dedekind zeta function

Dedekind ζ-function

Let E be a number field. For $\mathrm{Re}(s) > 1$

$$\zeta_E(s) = \sum_{I \neq 0 \text{ ideals in } \mathcal{O}_E} \frac{1}{N_{E/\mathbb{Q}}(I)^s}.$$

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Functional equation

Up to modifying $\zeta_E(s)$ by some Γ-factors, we get $\Lambda_E(s)$, which satisfies $\Lambda_E(s) = \Lambda_E(1 - s)$.

Analytic class number formula

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$$\lim_{s\to 1}(s-1)\zeta_E(s) = \frac{2^{r_1}(2\pi)^{r_2} h_E \mathrm{Reg}_E}{w_E |d_E|^{1/2}}$$

Here:

- h_E is the class number of E;
- w_E is the number of roots of unity in E;
- d_E is the discriminant of $E/\mathbb{Q}$;
- r_1, r_2 respectively the real and the pairs of complex embeddings of E;
- Reg_E is the regulator.

Analytic class number formula

Another formula

From the analytic class number formula and the functional equation one can show:

$$\lim_{s\to 0} \frac{\zeta_E(s)}{s^{r_1+r_2-1}} = -\frac{h_E \mathrm{Reg}_E}{w_E}.$$

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Let F/E be a Galois-extension of number fields with Galois group Γ, let $\chi \in \mathrm{Irr}(\Gamma)$ be an irreducible complex character of Γ, then one can define the Artin L-function $L(s, F/E, \chi)$.

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$$\zeta_F(s) = \prod_{\chi} L(s, F/E, \chi)^{\chi(1)}.$$

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Equivariant Artin L-function

If Γ is abelian we have an equivariant version:

$$\theta_{F/E}(s) = \sum_{\chi \in \hat{\Gamma}} L(s, F/E, \chi^{-1}) e_{\chi}.$$

Epsilon constants conjecture

Functional equation

Again one can modify $L(s, F/E, \chi)$ by some Γ-factors and obtain $\Lambda(s, F/E, \chi)$, which satisfies

$$\Lambda(s, F/E, \chi) = \varepsilon(s, F/E, \chi)\Lambda(1 - s, F/E, \bar{\chi}).$$

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Remark

The main building blocks of $\varepsilon(s, F/E, \chi)$ are the discriminant of F/E and a Gauß sum.

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Equivariant Tamagawa number conjecture

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Seminal work on the epsilon constant conjecture was done by Bloch-Kato and by Benois-Berger.

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Let

$$\rho^{\mathrm{nr}}\colon G_K \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$$

be an unramified representation of $G_K = \mathrm{Gal}(K^c/K)$. We will focus on the case $V = \mathbb{Q}_p^r(1)(\rho^{\mathrm{nr}})$, where the (1) stands for the twist with the cyclotomic character.

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Our formulation of the conjecture will be an equality in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$. For G abelian, this is isomorphic to $\mathbb{Q}_p[G]^\times/\mathbb{Z}_p[G]^\times$.

The main ingredients

The epsilon constants

$\varepsilon_D(N/K, V) \in Z(\mathbb{Q}_p[G])$ (the center of $\mathbb{Q}_p[G]$) is basically a Gauß sum (up to an extra factor).

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A sublattice

Let $T \subseteq V$ be a G_K-stable $\mathbb{Z}_p$-sublattice such that $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$.
In our case $T = \mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}})$.

A perfect complex

Theorem (C.)

Let $R\Gamma(N, T)$ be the complex of the G_N-invariants of the standard resolution of T. One can construct explicitly a bounded complex of cohomologically trivial G-modules which represents $R\Gamma(N, T)$. Its cohomology is:

1. $H^1(N, T) = (\prod_r \widehat{N_0^\times}(\rho^{\mathrm{nr}}))^{G_N}$, *where N_0 is the completion of the maximal unramified extension and the hat stands for the p-completion.*
2. $H^2(N, T) = \mathbb{Z}_p^r(\rho^{\mathrm{nr}})/(F_N - 1)\mathbb{Z}_p^r(\rho^{\mathrm{nr}})$,
3. $H^i(N, T) = 0$ *for* $i \neq 1, 2$.

The epsilon constant conjecture

The cohomological term

To a perfect complex (i.e. quasi-isomorphic to a bounded complex of f.g. projective $\mathbb{Z}_p[G]$-modules.) with a trivialisation, one can associate an Euler characteristic:

$$C_{N/K} = -\chi_{\mathbb{Z}_p[G],B_{\mathrm{dR}}[G]}(R\Gamma(N,T) \oplus \mathrm{Ind}_{N/\mathbb{Q}_p} T[0], \exp_V \circ \mathrm{comp}_V^{-1}).$$

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The epsilon constant conjecture

Some other terms are necessary:

$$R_{N/K} = C_{N/K} + U_{\mathrm{cris}} + rm\hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(t) - mU_{tw}(\rho^{\mathrm{nr}}_{\mathbb{Q}_p}) - rU_{N/K} + \hat{\partial}^1_{\mathbb{Z}_p[G], B_{\mathrm{dR}}[G]}(\varepsilon_D(N/K, V)).$$

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The conjecture $C^{na}_{EP}(N/K, V)$ states that $R_{N/K} = 0$.

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- Breuning, Bley-Debeerst: $[N : \mathbb{Q}_p]$ small.
- Bley-C: N/K weakly ramified and abelian, with cyclic ramification group, inertia degree coprime to $[K : \mathbb{Q}_p]$ and $K/\mathbb{Q}_p$ unramified.

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Remark

An Iwasawa theoretic version of the conjecture by A. Nickel, together with some work in progress of Burns-Nickel will give a new proof of the above results.

Higher dimensional results

Theorem (Bley-C.)

Let N/K be a tame extension of p-adic number fields and let

$$\rho^{\mathrm{nr}}_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$$

be an unramified representation of $G_{\mathbb{Q}_p}$. Let ρ^{nr} denote the restriction of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ to G_K. Then $C_{EP}^{na}(N/K, V)$ is true for N/K and $V = \mathbb{Q}_p^r(1)(\rho^{\mathrm{nr}})$, if $\det(\rho^{\mathrm{nr}}(F_N) - 1) \neq 0$.

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Remark

Recall: $H^2(N, T) = \mathbb{Z}_p^r(\rho^{\mathrm{nr}})/(F_N - 1)\mathbb{Z}_p^r(\rho^{\mathrm{nr}})$. The condition $\det(\rho^{\mathrm{nr}}(F_N) - 1) \neq 0$ holds, if and only if $H^2(N, \mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}}))$ is finite.

Higher dimensional results

Theorem (Bley-C.)

Let $K/\mathbb{Q}_p$ be unramified of degree m and let N/K be weakly and wildly ramified, finite and abelian with cyclic ramification group. Let d be the inertia degree of N/K, let $\tilde{d}$ denote the order of $\rho^{\mathrm{nr}}(F_N)$ mod p in $\mathrm{Gl}_r(\mathbb{Z}_p/p\mathbb{Z}_p)$ and assume that m and d are relatively prime. Let $\rho^{\mathrm{nr}}_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \longrightarrow \mathrm{Gl}_r(\mathbb{Z}_p)$ be an unramified representation of $G_{\mathbb{Q}_p}$ and let ρ^{nr} denote the restriction of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$ to G_K. Assume that $\det(\rho^{\mathrm{nr}}(F_N) - 1) \neq 0$ and, in addition, that one of the following three conditions holds:

(a) $\rho^{\mathrm{nr}}(F_N) - 1$ *is invertible modulo p;*

(b) $\rho^{\mathrm{nr}}(F_N) \equiv 1 \pmod{p}$*;*

(c) $\gcd(\tilde{d}, m) = 1$ *and* $\det(\rho^{\mathrm{nr}}(F_N)^{\tilde{d}} - 1) \neq 0$.

Then $C_{EP}^{na}(N/K, V)$ is true for N/K and $V = \mathbb{Q}_p^r(1)(\rho^{\mathrm{nr}})$.

Some geometry

Final remark

If $A/\mathbb{Q}_p$ is an abelian variety of dimension r with good ordinary reduction, then the Tate module of the associated formal group $\hat{A}$ is isomorphic to $\mathbb{Z}_p^r(1)(\rho^{\mathrm{nr}}_{\mathbb{Q}_p})$ for an appropriate choice of $\rho^{\mathrm{nr}}_{\mathbb{Q}_p}$. By a result of Mazur $\det(\rho^{\mathrm{nr}}(F_L) - 1) \neq 0$ is automatically satisfied for any finite extension $L/\mathbb{Q}_p$.

Thank you for your attention!