The epsilon constant conjecture for higher dimensional unramified twists of $\mathbb{Z}_p^r(1)$

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Let $E$ be a number field. For $\text{Re}(s) > 1$

$$\zeta_E(s) = \sum_{I \neq 0 \text{ ideals in } \mathcal{O}_E} \frac{1}{N_{E/Q}(I)^s}.$$ 

This is extended analytically and has a pole at $s = 1$. 
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Dedekind $\zeta$-function

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Functional equation

Up to modifying $\zeta_E(s)$ by some $\Gamma$-factors, we get $\Lambda_E(s)$, which satisfies $\Lambda_E(s) = \Lambda_E(1 - s)$. 
Motivation

The epsilon constant conjecture

Analytic class number formula

\[
\lim_{s \to 1} (s - 1) \zeta_E(s) = \frac{2^{r_1} (2\pi)^{r_2} h_E \text{Reg}_E}{w_E |d_E|^{1/2}}
\]

Here:

- \( h_E \) is the class number of \( E \);
- \( w_E \) is the number of roots of unity in \( E \);
- \( d_E \) is the discriminant of \( E/\mathbb{Q} \);
- \( r_1, r_2 \) respectively the real and the pairs of complex embeddings of \( E \);
- \( \text{Reg}_E \) is the regulator.
Another formula

From the analytic class number formula and the functional equation one can show:

$$\lim_{s \to 0} \frac{\zeta_E(s)}{s^{r_1+r_2-1}} = -\frac{h_E \operatorname{Reg}_E}{w_E}.$$
Artin L-functions

Let $F/E$ be a Galois-extension of number fields with Galois group $\Gamma$, let $\chi \in \text{Irr}(\Gamma)$ be an irreducible complex character of $\Gamma$, then one can define the Artin L-function $L(s, F/E, \chi)$. 

Relation to $\zeta_N(s)$

$\zeta_F(s) = \prod_{\chi} L(s, F/E, \chi) \chi(1)$. 

Equivariant Artin L-function

If $\Gamma$ is abelian we have an equivariant version: 

$\theta_{F/E}(s) = \sum_{\chi \in \hat{\Gamma}} L(s, F/E, \chi^{-1}) e^{\chi}$. 

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Epsilon constants conjecture

**Functional equation**

Again one can modify $L(s, F/E, \chi)$ by some $\Gamma$-factors and obtain $\Lambda(s, F/E, \chi)$, which satisfies

$$\Lambda(s, F/E, \chi) = \varepsilon(s, F/E, \chi)\Lambda(1 - s, F/E, \bar{\chi}).$$
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**Remark**

The main building blocks of $\varepsilon(s, F/E, \chi)$ are the discriminant of $F/E$ and a Gauß sum.
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- ETNC(0)
- ETNC(1).
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Seminal work on the epsilon constant conjecture was done by Bloch-Kato and by Benois-Berger.
The epsilon constant conjecture

The setting

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$$\rho^{nr} : G_K \longrightarrow \text{Gl}_r(\mathbb{Z}_p)$$

be an unramified representation of $G_K = \text{Gal}(K^c/K)$. We will focus on the case $V = \mathbb{Q}_p^r(1)(\rho^{nr})$, where the (1) stands for the twist with the cyclotomic character.
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Our formulation of the conjecture will be an equality in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$. For $G$ abelian, this is isomorphic to $\mathbb{Q}_p[G]^\times / \mathbb{Z}_p[G]^\times$. 
The main ingredients

The epsilon constants

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A sublattice

Let \( T \subseteq V \) be a \( G_K \)-stable \( \mathbb{Z}_p \)-sublattice such that \( V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T \).

In our case \( T = \mathbb{Z}_p^r(1)(\rho^{nr}) \).
A perfect complex

Theorem (C.)

Let $R\Gamma(N, T)$ be the complex of the $G_N$-invariants of the standard resolution of $T$. One can construct explicitly a bounded complex of cohomologically trivial $G$-modules which represents $R\Gamma(N, T)$. Its cohomology is:

1. $H^1(N, T) = (\prod_r \hat{N_0}^{\times}(\rho_{nr}))^{G_N}$, where $N_0$ is the completion of the maximal unramified extension and the hat stands for the $p$-completion.
2. $H^2(N, T) = \mathbb{Z}_p(\rho_{nr})/(F_N - 1)\mathbb{Z}_p(\rho_{nr})$,
3. $H^i(N, T) = 0$ for $i \neq 1, 2$. 
The epsilon constant conjecture

The cohomological term

To a perfect complex (i.e. quasi-isomorphic to a bounded complex of f.g. projective \( \mathbb{Z}_p[G] \)-modules.) with a trivialisation, one can associate an Euler characteristic:

\[
C_{N/K} = -\chi_{\mathbb{Z}_p[G], B_{dR}[G]}(R\Gamma(N, T) \oplus \text{Ind}_{N/\mathbb{Q}_p} T[0], \exp_V \circ \text{comp}_V^{-1}).
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Some other terms are necessary:

$$R_{N/K} = C_{N/K} + U_{\text{cris}} + rm\hat{\delta}^1_{\mathbb{Z}_p[G], B_{dR}[G]}(t) - mU_{tw}(\rho_{\mathbb{Q}_p}^{nr}) - rU_{N/K} + \hat{\delta}^1_{\mathbb{Z}_p[G], B_{dR}[G]}(\varepsilon_D(N/K, V)).$$
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The conjecture $C_{EP}^{na}(N/K, V)$ states that $R_{N/K} = 0$. 
Results for $V = \mathbb{Q}_p(1)$

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- Breuning, Bley-Debeerst: $[N : \mathbb{Q}_p]$ small.
- Bley-C: $N/K$ weakly ramified and abelian, with cyclic ramification group, inertia degree coprime to $[K : \mathbb{Q}_p]$ and $K/\mathbb{Q}_p$ unramified.
Results for unramified twists of $\mathbb{Q}_p(1)$

Let $V = \mathbb{Q}_p(\chi^{nr})(1)$, where $\chi^{nr}$ is an unramified character of $G_K$, which is the restriction of an unramified character of $G_{\mathbb{Q}_p}$. 
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Remark

An Iwasawa theoretic version of the conjecture by A. Nickel, together with some work in progress of Burns-Nickel will give a new proof of the above results.
Higher dimensional results

Theorem (Bley-C.)

Let \( N/K \) be a tame extension of \( p \)-adic number fields and let

\[
\rho_{Q_p}^{nr} : G_{Q_p} \longrightarrow \text{Gl}_r(\mathbb{Z}_p)
\]

be an unramified representation of \( G_{Q_p} \). Let \( \rho^{nr} \) denote the restriction of \( \rho_{Q_p}^{nr} \) to \( G_K \). Then \( C_{EP}^{na}(N/K, V) \) is true for \( N/K \) and \( V = Q_p^r(1)(\rho^{nr}) \), if \( \det(\rho^{nr}(F_N) - 1) \neq 0 \).
Theepsilon-constant conjecture

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Let $N/K$ be a tame extension of $p$-adic number fields and let

$$\rho_{Q_p}^{nr}: G_{Q_p} \rightarrow \text{Gl}_r(\mathbb{Z}_p)$$

be an unramified representation of $G_{Q_p}$. Let $\rho^{nr}$ denote the restriction of $\rho_{Q_p}^{nr}$ to $G_K$. Then $C^n_{EP}(N/K, V)$ is true for $N/K$ and $V = \mathbb{Q}_p^r(1)(\rho^{nr})$, if $\det(\rho^{nr}(F_N) - 1) \neq 0$.

**Remark**

Recall: $H^2(N, T) = \mathbb{Z}_p^r(\rho^{nr})/(F_N - 1)\mathbb{Z}_p^r(\rho^{nr})$. The condition $\det(\rho^{nr}(F_N) - 1) \neq 0$ holds, if and only if $H^2(N, \mathbb{Z}_p^r(1)(\rho^{nr}))$ is finite.
**Theorem (Bley-C.)**

Let $K/\mathbb{Q}_p$ be unramified of degree $m$ and let $N/K$ be weakly and wildly ramified, finite and abelian with cyclic ramification group. Let $d$ be the inertia degree of $N/K$, let $\tilde{d}$ denote the order of $\rho^{nr}(F_N) \mod p$ in $\text{Gl}_r(\mathbb{Z}_p/p\mathbb{Z}_p)$ and assume that $m$ and $d$ are relatively prime. Let $\rho^{nr}_{\mathbb{Q}_p} : G_{\mathbb{Q}_p} \to \text{Gl}_r(\mathbb{Z}_p)$ be an unramified representation of $G_{\mathbb{Q}_p}$ and let $\rho^{nr}$ denote the restriction of $\rho^{nr}_{\mathbb{Q}_p}$ to $G_K$. Assume that $\det(\rho^{nr}(F_N) - 1) \neq 0$ and, in addition, that one of the following three conditions holds:

(a) $\rho^{nr}(F_N) - 1$ is invertible modulo $p$;

(b) $\rho^{nr}(F_N) \equiv 1 \pmod{p}$;

(c) $\gcd(\tilde{d}, m) = 1$ and $\det(\rho^{nr}(F_N)^{\tilde{d}} - 1) \neq 0$.

Then $C^{na}_{EP}(N/K, V)$ is true for $N/K$ and $V = \mathbb{Q}_p^r(1)(\rho^{nr})$. 
Final remark

If $A/\mathbb{Q}_p$ is an abelian variety of dimension $r$ with good ordinary reduction, then the Tate module of the associated formal group $\hat{A}$ is isomorphic to $\mathbb{Z}_p^r(1)(\rho_{\mathbb{Q}_p}^{nr})$ for an appropriate choice of $\rho_{\mathbb{Q}_p}^{nr}$. By a result of Mazur, $\det(\rho^{nr}(F_L) - 1) \neq 0$ is automatically satisfied for any finite extension $L/\mathbb{Q}_p$. 
Thank you for your attention!