# Boundary unique continuation on $C^1$-Dini domains

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# Background

### Motivation

Let $u$ be a harmonic function in $D \subset \mathbb{R}^d$.

- Let $d = 2$. If $\nabla u$ vanishes on a boundary set $E \subset \partial D$ with positive measure, then $u \equiv const$.

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### Motivation

Let $u$ be a harmonic function in $D \subset \mathbb{R}^d$.

- Let $d = 2$. If $\nabla u$ vanishes on a boundary set $E \subset \partial D$ with positive measure, then $u \equiv const$.
- (A classical question originating from Bers) Let $d \geq 3$. Is it possible that $u$ and $\nabla u$ vanish on a boundary set $E \subset \partial D$ with positive surface measure, i.e.

$$\mathcal{H}^{d-1}(\{x \in \partial D : u(x) = 0 = |\nabla u(x)|\}) > 0?$$

Theorem (Bourgain-Wolff 1990)

*When* $d \geq 3$*, there exists a non-trivial harmonic function* $u \in C^1(\overline{\mathbb{R}^d_+})$ *such that*

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**Question:** Assuming that $u \equiv 0$ on an open set $U \subset \partial D$, how big can the set $\{x \in \partial D : \nabla u(x) = 0\}$ be?

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**Theorem (Tolsa 2020, Adolfsson-Escauriaza 1997)**

*Let $D$ be a Lipschitz domain with sufficiently small Lipschitz constant. Let $u \in C(\overline{D})$ be a non-trivial harmonic function in $D$. Suppose that $u \equiv 0$ on $\partial D \cap B_{5R}$. Then*

$$\mathcal{H}^{d-1}(\{x \in \partial D \cap B_R : \nabla u(x) = 0\}) = 0.$$

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- Moreover, suppose the *frequency function* of $u$ satisfies $N_0(4R) \leq \Lambda$.

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Theorem (Kenig-Z 2021)

*We have the following bound on the size of* $\mathcal{S}(u)$

$$\mathcal{H}^{d-2}\left(\mathcal{S}(u) \cap B_R(0)\right) \leq \mathcal{M}^{d-2,*}\left(\mathcal{S}(u) \cap B_R(0)\right) \leq C(\Lambda),$$

*and* $\mathcal{S}(u) \cap B_R(0)$ *is* $(d-2)$*-rectifiable.*

### Definition

We say a domain $D \subset \mathbb{R}^d$ is a $C^1$-**Dini domain** if locally, it is above the graph of a $C^1$ function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$, i.e.

$$D = \left\{ (x, x_d) \in \mathbb{R}^d : x_d > \varphi(x) \right\},$$

where $\varphi$ satisfies $|\nabla\varphi(x) - \nabla\varphi(y)| \leq \theta(|x - y|)$ and $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function satisfying the Dini condition

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## Remark

- In particular $C^{1,\alpha}$ domains ($0 < \alpha < 1$) are Dini domains.
- Sharp for $u \in C^1(\overline{D})$.

# Analogous result in the interior

**Theorem (Naber-Valtorta 2018)**

*Let $u$ be a non-trivial harmonic function in $B_3(0) \subset \mathbb{R}^d$. Suppose the frequency function of $u$ satisfies $N_0(2) \le \Lambda$. Let*

$$\mathcal{C}(u) = \{x \in D : \nabla u(x) = 0\}$$

*be the critical set of $u$. Then $\mathcal{C}(u) \cap B_1(0)$ is $(d-2)$-rectifiable, and*

$$\mathcal{H}^{d-2}(\mathcal{C}(u) \cap B_1(0)) \le C(\Lambda).$$

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# Frequency function

Assume WOLG that $u(0) = 0$. We define the frequency function centered at 0 to be

$$r \mapsto N(r) := \frac{rD(r)}{H(r)} = \frac{r \iint_{B_r(0)} |\nabla u|^2 \, dx}{\int_{\partial B_r(0)} u^2 \, d\mathcal{H}^{d-1}}.$$

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**Example**

Suppose $u = P_{N_0}$ is a homogeneous harmonic polynomial of degree $N_0$. Then $N(r) \equiv N_0$.

**Proposition (Monotonicity formula of the frequency function)**

*The frequency function satisfies*

$$\frac{d}{dr}N(r) = \frac{2r}{H(r)} \int_{\partial B_r(0)} \left| \partial_r u - \frac{N(r)}{r} u \right|^2 \, d\mathcal{H}^{d-1} \geq 0,$$

*where $\partial_r u$ denotes the radial derivative of $u$.*
*Thus $r \mapsto N(r)$ is monotone increasing, and the limit $\lim_{r \to 0+} N(r)$ exists.*

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## Remark

- $N := \lim_{r\to 0+} N(r)$ is the degree of the leading order term in the expansion of $u(x)$ near 0.
- The assumption $N_0(2) \leq \Lambda$ means the growth of the harmonic function $u$ can not be too fast near 0.

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- $0 \in \mathcal{C}(u)$ (namely $\nabla u(0) = 0$) $\iff N \geq 2$.
- $N = 1$, i.e. $P_N$ is linear $\iff P_N$ is invariant in $(d-1)$ linearly independent directions.
- $N \geq 2 \iff P_N$ is invariant in at most $(d-2)$ linearly independent directions.

### Observation II: Cone splitting

Let $h$ be a non-trivial harmonic function in $\mathbb{R}^d$, and $x_1, x_2 \in \mathbb{R}^d$. Suppose $h$ is homogeneous of degree $N_i$ with respect to $x_i$, for $i = 1, 2$. Then $N_1 = N_2 \in \mathbb{N}$, and $h$ is invariant along the direction $x_2 - x_1$, i.e.

$$h(y + t(x_2 - x_1)) = h(y), \quad \text{for any } y \in \mathbb{R}^d \text{ and } t \in \mathbb{R}.$$

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- If $N_x(r) = N_x(r/2)$, then $u =$ (a constant multiple of) some hhP $P_N$.
- If $N_x(r) - N_x(r/2) \leq \delta$, then $u$ is $\epsilon$-close to (a constant multiple of) some hhP $P_N$ in $B_r(x)$.
- If $x, x' \in \mathbb{R}^d$ are two distinct points such that $|x - x'| < r/2$, and

$$N_x(r) - N_x(r/2) \leq \delta,$$

$$N_{x'}(r) - N_{x'}(r/2) \leq \delta,$$

then $P^x$ is almost invariant along the direction $\frac{x'-x}{r}$.

Thank you!