Colourful components in $k$-caterpillars and planar graphs

Janka Chlebíková,¹ Clément Dallard²
June 22nd 2021

¹ University of Portsmouth, UK
² FAMNIT, University of Primorska, Slovenia
We consider undirected graphs with colored vertices.
We consider undirected graphs with colored vertices.

**Some definitions**

A *colourful component* is a *connected component* whose vertices have different colours.
We consider undirected graphs with colored vertices.

**Some definitions**

A *colourful component* is a *connected component* whose vertices have different colours.

A graph is *colourful* if all its connected components are colourful.
We consider undirected graphs with colored vertices.

**Some definitions**

A *colourful component* is a *connected component* whose vertices have different colours.

A graph is *colourful* if all its connected components are colourful.

A *bad path* is a path with endpoints of the same colour.
Our problems

- **Colourful Components:**
  Are there at most $p$ edges whose removal makes the graph colourful?

- **Colourful Partition:**
  Is there a partition of the graph with at most $p$ parts such that each part is a colourful component?
Our problems

- **Colourful Components:**
  Are there at most $p$ edges whose removal makes the graph colourful?

- **Colourful Partition:**
  Is there a partition of the graph with at most $p$ parts such that each part is a colourful component?

**Motivation**
Partition a set of genes in *orthologous genes*, which are *sets of genes from different species* that have evolved through speciation events only.
Our problems

- **Colourful Components:** Are there at most $p$ edges whose removal makes the graph colourful?

- **Colourful Partition:** Is there a partition of the graph with at most $p$ parts such that each part is a colourful component?

Observation

*Colourful Components* and *Colourful Partition* are linear-time equivalent on forests.
Our problems

- **Colourful Components**: Are there at most $p$ edges whose removal makes the graph colourful?

- **Colourful Partition**: Is there a partition of the graph with at most $p$ parts such that each part is a colourful component?

Observation

*Colourful Components* and *Colourful Partition* are linear-time equivalent on forests.

We can only consider connected graphs.
A $k$-caterpillar is a tree in which all the vertices are within distance $k$ of a central path, called the backbone.
Known results on trees with bounded diameter

Example of a tree with diameter 4.

Theorem (Bruckner et al., ’12)

*Colourful Components* is NP-complete on trees with diameter 4.
Known results on trees with bounded diameter

Example of a tree with diameter 4.

**Theorem (Bruckner et al., ’12)**

*COLOURFUL COMPONENTS* is NP-complete on trees with diameter 4.

**Corollary**

*COLOURFUL COMPONENTS* is NP-complete on 2-caterpillars.
Theorem (Dondi and Sikora, ’18)

Colourful Components is solvable in time $O(n^2)$ on paths with $n$ vertices (trees with max. degree $\leq 2$).
Theorem (Dondi and Sikora, ’18)

*COLOURFUL COMPONENTS* is solvable in time $O(n^2)$ on paths with $n$ vertices (trees with max. degree $\leq 2$).

Theorem (Bulteau et al., ’19)

*COLOURFUL COMPONENTS* is NP-complete on trees with maximum degree 6.
Known results on trees with bounded degree

Theorem (Dondi and Sikora, ’18)

**COLOURFUL COMPONENTS** is solvable in time $O(n^2)$ on paths with $n$ vertices (trees with max. degree $\leq 2$).

Theorem (Bulteau et al., ’19)

**COLOURFUL COMPONENTS** is NP-complete on trees with maximum degree 6.

The complexity of **COLOURFUL COMPONENTS** on trees with maximum degree $\leq 5$ was left open in the same paper.
Our goal

We would like complexity dichotomies for \textsc{Colourful Components} on trees with respect to:

- the maximum degree $d$ of the input tree,
Our goal

We would like complexity dichotomies for Colourful Components on trees with respect to:

• the maximum degree $d$ of the input tree,
• the smallest integer $k$ such that the input tree is a $k$-caterpillar,
Our goal

We would like complexity dichotomies for Colourful Components on trees with respect to:

- the maximum degree $d$ of the input tree,
- the smallest integer $k$ such that the input tree is a $k$-caterpillar,
- both $d$ and $k$. 
Bad news!

**Theorem (Chlebíková and D.)**

*Colourful Components* is NP-complete on:

- 4-caterpillars with maximum degree $\leq 3$,
- 3-caterpillars with maximum degree $\leq 4$, and
- 2-caterpillars with maximum degree $\leq 5$.

**Reduction from 3,3-SAT** $(\leq 3\text{ var. per clause, } \leq 3\text{ occurrences var.})$. 

---

(a) Variable gadget of $x_1$
(b) Clause gadget of type A of the clause $C_3$
(c) Clause gadget of type B of the clause $C_2$
(d) Clause gadget of type C of the clause $C_2$
Bad news!

**Theorem (Chlebíková and D.)**

*Colourful Components* is NP-complete on:

- 4-caterpillars with maximum degree $\leq 3$,
- 3-caterpillars with maximum degree $\leq 4$, and
- 2-caterpillars with maximum degree $\leq 5$.

**Variable gadget of $x_1$**

- $r_{x_1}$
- $x_1$
- $x_{1,2}$
- $x_{1,5}$
- $\bar{x}_{1,3}$

**Clause gadget of $C_2$ of type B of the clause $C_2$**

- $r_{C_2}$
- $z_2$
- $y_2$
- $\ell_{1,2}$

- $y_2'$
- $\ell_{2,2}$

- $z_2'$
- $\ell_{3,2}$
Definition

A cyclic $k$-caterpillar is a connected graph $G$ with a unique cycle $B$, called the backbone, such that for any $e \in E(B)$, the graph $G - e$ is a $k$-caterpillar.

A $k$-pseudocaterpillar is either a $k$-caterpillar or a cyclic $k$-caterpillar.
But pseudocaterpillars are fine

**Definition**

A *cyclic $k$-caterpillar* is a connected graph $G$ with a unique cycle $B$, called the backbone, such that for any $e \in E(B)$, the graph $G - e$ is a $k$-caterpillar.

A *$k$-pseudocaterpillar* is either a $k$-caterpillar or a cyclic $k$-caterpillar.

**Theorem (Chlebíková and D.)**

*Colours* *Components* is can be solved in linear time on 1-pseudocaterpillars.
We can see the graph as a collection of stars that are connected via their internal vertices (in a path or cycle manner).
Algorithm’s idea

We preprocess the graph, store the removed edges and focus on the component containing the backbone.
We obtain that every star is colourful.
We can show that there exists an optimal solution (a set of edges) for which every edge belongs to the backbone.
Algorithm’s idea

It is in fact enough to consider a specific kind of bad paths: the critical bad paths.
Algorithm’s idea

We represent the *critical bad paths* as arcs on the circle and compute in linear time (Hsu and Tsai, ’91) a minimum clique cover on the circular-arc graph.
Algorithm’s idea

For each clique in the clique cover, we remove one corresponding edge in the backbone. These edges intersect every critical bad path.
Eventually, we return the union of the edges removed in the preprocessing and the newly obtained. This is an optimal solution.
Recap

- NP-complete on 4-caterpillars with max. degree $\leq 3$.
- NP-complete on 3-caterpillars with max. degree $\leq 4$.
- NP-complete on 2-caterpillars with max. degree $\leq 5$.
- Linear-time solvable on 1-pseudocaterpillars.
Recap

- NP-complete on 4-caterpillars with max. degree $\leq 3$.
- NP-complete on 3-caterpillars with max. degree $\leq 4$.
- NP-complete on 2-caterpillars with max. degree $\leq 5$.
- Linear-time solvable on 1-pseudocaterpillars.

So we obtain two complexity dichotomies on trees:

- **Colourful Components** is linear-time solvable on trees with maximum degree $d$ if $d \leq 2$, and NP-complete otherwise.
- **Colourful Components** is linear-time solvable on $k$-caterpillars if $k \leq 1$, and NP-complete otherwise.
A complexity dichotomy in terms of $d$ and $k$?

The complexity of **Colourful Components** is still open on:

- 2-caterpillars with max. degree $\leq 4$,
- 2-caterpillars with max. degree $\leq 3$,
- 3-caterpillars with max. degree $\leq 3$. 
Known results on planar graph with bounded degree

Theorem (Bruckner et al., ’12)

*COLOURFUL COMPONENTS* is NP-complete on 3-coloured planar graphs with maximum degree 6.
Known results on planar graph with bounded degree

Theorem (Bruckner et al., '12)

**Colourful Components** is NP-complete on 3-coloured planar graphs with maximum degree 6.

Question: Can the maximum degree be decreased while preserving NP-completeness (and bounded number of colours)?
Theorem (Chlebíková and D.)

**COLOURFUL COMPONENTS** is NP-complete on 5-coloured planar graphs with maximum degree 4 and on 12-coloured planar graphs with maximum degree 3.

Gadget with vertices of degree 4.  
Gadget with vertices of degree 3.

This time, we reduce from PLANAR 3-SAT.
Planar graphs with even smaller degrees (but more colours)

Theorem (Chlebíková and D.)

**COLOURFUL COMPONENTS** is NP-complete on 5-coloured planar graphs with maximum degree 4 and on 12-coloured planar graphs with maximum degree 3.

There might still be room for improvement...

Eventually, we would like a complexity dichotomy based on the maximum degree and number of colours (in planar graphs).
Thank you!