Digital Collections of Examples in Mathematical Sciences

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23 June 2021
Plan of Talk

1. Important Collections in Pure Mathematics
2. Important Test Suites in SAT/SMT
3. The Lack of Test Suites elsewhere
4. Way Forward?
Data Citation

- Is a mess in practice [vdSNI+19]: only 1.16% of dataset DOIs in Zenodo are cited (and 98.5% of these are self-citation).
- Is poorly harvested: [vdSNI+19, Figure 5].

There are between 4,000–20,000 data sets waiting to be cited.
Data Citation

- Is a mess in practice [vdSNI+19]: only 1.16% of dataset DOIs in Zenodo are cited (and 98.5% of these are self-citation).
- Is poorly harvested: [vdSNI+19, Figure 5].
- Is still a subject of some uncertainty: [MN12, KS14]
- Changes are still being proposed [DPS+20]
- *de facto* people cite a paper if they can find one.
Important Databases in Pure Mathematics

**OEIS**
Online Encyclopedia of Integer Sequences [Slo03];


⚠️ But you have to search the website to find it!
Group Theory (as an example)

- The Classification of Finite Simple Groups
- The Transitive Groups acting on $n$ points: [BM83] ($n \leq 11$); [Roy87] ($n = 12$); [But93] ($n = 14, 15$); [Hul96] ($n = 16$); [Hul05] ($17 \leq n \leq 31$); [CH08] ($n = 32$).
- These are in GAP (and MAGMA), except that $n = 32$ isn’t in the default build.
  - These are a really great resource (if that’s what you want)
  - How do you cite them? “[The21, GAP transgrp library]”? 

Also
- Other libraries such as primitive groups
- Group Theory is “easy”: for a given $n$ there are a finite number and we “just” have to list them.
SAT Solving

SAT solving, normally seen as solving a Boolean expression written in CNF. Given a 3-literals/clause CNF satisfiability problem,

\[(l_{1,1} \lor l_{1,2} \lor l_{1,3}) \land (l_{2,1} \lor l_{2,2} \lor l_{2,3}) \land \cdots \land (l_{N,1} \lor l_{N,2} \lor l_{N,3}),\]

Clause 1

where \(l_{i,j} \in \{x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots\}\), is it satisfiable? In other words, is there an assignment of \(\{T, F\}\) to the \(x_i\) such that all the clauses are simultaneously true.

3-SAT: the quintessential NP-complete problem [Coo66]. 2-SAT is polynomial, and \(k\)-SAT for \(k > 3\) is polynomial-transformable into 3-SAT. In practice we deal with SAT — i.e. no limitations on the length of the clauses.

Let \(n\) be the number of \(i\) such that \(x_i\) (and/or \(\overline{x_i}\)) actually occur. Typically \(n\) is of a similar size to \(N\).
Despite being NP-complete, nearly all examples are easy (e.g. [KS00]),
either easily solved (SAT) or easily proved insoluble (UNSAT) and
for random problems there seems to be a distinct phase transition
between the two: [GW94, AP04, AP06].
This means that constructing difficult examples is itself difficult,
and a research area in itself: [Spe15, BC18].
SAT solving has many applications, so we want effective solvers for
“real” problems, not just “random” ones.
Fundamental question: what does this mean?
SAT Contests: [http://www.satcompetition.org](http://www.satcompetition.org)

 Been run since 2002. In the early years, distinct tracks for Industrial/Handmade/Random problems: this has been abandoned. The methodology is that the organisers accept submissions (from contestants and others), then produce a list of problems (in a standard format) and set a time (and memory) limit, and see how many of the problems the submitted systems can solve on the contest hardware.

 SAT is easy to certify (just produce a list of values), UNSAT is much harder, but since 2013 the contest has required proofs of UNSAT for the UNSAT track, and since 2020 in all tracks, in DRAT: a specified format (some have been > 100GB). The general feeling is that these contests have really pushed the development of SAT solvers, roughly speaking ×2/year. For comparison, Linear Programming has done ×1.8 over a greater timeline [Bix15].
SMT: life beyond SAT

Consider a theory $T$, with variables $y_j$, and various Boolean-valued statements in $T$ of the form $F_i(y_1, \ldots, y_n)$, and a CNF with $F_i(y_1, \ldots, y_n)$ rather than just $x_i$. Then the SAT/UNSAT question is similar ($\exists$ values of $y_i \ldots$), and the community runs SMT Competitions (https://smt-comp.github.io/2020/), but a separate track for each theory, as the problems will be different. The SMTLIB format [BFT17] provides a standard input format. UNSAT is in general unsolved (but see [KAED21] for one example). There is substantial progress in SMT-solving over the years, possibly similar to SAT.
Obviously, Group Theory (etc.) are part of computer algebra: what about the rest?

In general the problems have a bad worst-case complexity, and we want effective solvers for “real” problems, not just “random” ones. The question is “what does this mean?”.

**Format**  No common standard. We do have OpenMath [BCC+17], but it’s not as widely supported as we would like.

**Contests**  None. Could SIGSAM organise them?

**Problem Sets**  No independent ones. Each author chooses his own.

**Archive**  Not really.
Polynomial g.c.d.

- NP-hard (for sparse polynomials, even univariate) [Pla84].
- Can be challenging for multivariates
  - No standard database: trawl previous papers (and often need to ask the authors)
- Verification is a challenge: one can check that the result is a \textit{common divisor}, but verifying \textit{greatest} is still NP-hard [Pla84].
Polynomial Factorisation

- Polynomial-time for dense encodings [LLL82], presumably NP-hard for sparse.
- No standard database: trawl previous papers (and often need to ask the authors)

Verification is a challenge: one can check that the result is a factorisation, but checking completeness (i.e. that these factors are irreducible) seems to be as hard as the original problem.

? With probability 1, a random polynomial is irreducible, so what are the interesting problems?
Gröbner Bases

- Doubly exponential (w.r.t. $n$) worst-case complexity [MR13], even if a prime ideal [Chi09].
+ There is a collection [BM96]
  - Very old (1996) and completely static.
  - Some examples only in PDF.
? No concept of UNSAT, but it’s not clear what a certificate might mean.
Real Algebraic Geometry (CAD)

- Doubly exponential (w.r.t. \( n \)) worst-case complexity [BD07]
  - There is a collection [Wil14]
  - Somewhat old (2014) and completely static.

- The DEWCAD project [BDE+21] might update this, but still issues of long-term conservation.

- Format: text, Maple and QEPCAD
- No concept of UNSAT (but see [KAED21]), but it’s not clear what a certificate might mean.
Integration

- Complexity is essentially unknown (but certainly involves g.c.d., factorisation etc.)
- A new question here is the “niceness” of the output.
- “Paper” mathematics produced large databases, e.g. [GR07].
  - PDF, and the devil to scan.
- Current best database is described in [JR10].
- Algorithm-based software (e.g. [Dav81]) has an internal proof of UNSAT, but I know of no software that can exhibit it.
Conclusions

1. The field of computer algebra really ought to invest in the sort of contests that have stimulated the SAT and SMT worlds.
2. This requires much larger databases of “relevant” problems than we currently have, and they need to be properly curated.
   + Technology, e.g. wikis, or GitHub, has greatly advanced since [BM96].
3. This would allow much better benchmarking technology [BDG17].
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