Alternating links, rational balls & tilings

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(on joint work with Josh Greene)
Questions

\[ S^3 = \emptyset B^4 \]

\[ \Sigma_2(S^3, \text{unknot}) \quad \Sigma_2(B^4, \text{unknotted disk}) \]

Generalize:

- Which rational homology 3-spheres \( Y \) bound rational homology 4-balls \( W \)?
- Which knots in \( S^3 \) are slice?
- Which links in \( S^3 \) bound surfaces \( F \hookrightarrow B^4, \chi(F) = 1 \)?

\[ \{ W = \Sigma_2(B^4,F) \} \text{ is \( \mathcal{QHB} \)} \]

[Everything smooth]

**Goal:** classify alternating links with \( \Sigma_2(S^3,L) = \emptyset \) \( \mathcal{QHB} \)
2) Example

L alternating link

Λb “black lattice” has
Gram matrix \[
\begin{bmatrix}
5 & 2 \\
2 & 4
\end{bmatrix},
\text{ rank 2, det } 16 = 4^2
\]

L bounds \( F = \text{disk} \sqcup \text{M"ob. band} \)

\( \chi(F) = 1 \)

\( \Sigma_2(S^3,L) = \partial \Sigma_2(B^4,F) \neq \emptyset \)
Theorem (Greene-O.)

Let $L$ be an alternating link s.t. $\Lambda_b$ has rank $n$ and determinant $2^n$.

TFAE:  
1. $\Sigma_2(S^3, L)$ bounds a $QHB$;
2. $\Lambda_b$ is a 2-cube tiling lattice;
3. $L$ is expanded from the crossingless unknot by a finite sequence of moves I and II;
4. $L$ may be converted to the $k$-component unlink by a sequence of $(k-1)$ band moves and finitely many Conway mutations for some $k \in \mathbb{N}$. 
2-cube tiling lattice:

A lattice \( \Lambda \subseteq \mathbb{Z}^n \) whose vertices are centres of cubes of side 2 which tile \( \mathbb{R}^n \).

Conway mutation:

\[
\text{tangle} \quad \rightarrow \quad \text{tangle}
\]

(suffices for our needs)
Moves I and II:

(Require $u, v$ in different components of black graph $\setminus \text{3w3}$)
Lots of examples

Black graph of unknot

moves I

& II

Replace any edge
with
Proof sketch

1. $\Sigma_2(S^3, L)$ bounds a QHB;
   
   Donaldson's diagonalisation then + Heegaard Floer correction terms (à la Greene-Jabuka)

2. $\Lambda_b$ is a 2-cube tiling lattice;

3. $L$ is expanded from the crossingless unknot by a finite sequence of moves I and II;

4. $L$ may be converted to the $k$-component unlink by a sequence of ($k-1$) band moves and finitely many Conway mutations for some $k \in \mathbb{N}$. 
8 Proof sketch

(1) $\Sigma_2(S^3,L)$ bounds a QHB;

(2) $\Lambda_b$ is a 2-cube tiling lattice; uses Minkowski conjecture (1896) proved by Hajós (1941): every tiling of $\mathbb{R}^n$ by cubes has a pair of cubes which share a facet.

(3) $L$ is expanded from the crossingless unknot $\bigcirc$ by a finite sequence of moves I and II;

(4) $L$ may be converted to the $k$-component unlink by a sequence of $(k-1)$ braid moves and finitely many Conway mutations for some $k \in \mathbb{N}$. 

\[ \]
Proof sketch

1. $\Sigma_2(S^3, L)$ bounds a $Q$H$B$;
2. $\Lambda_b$ is a 2-cube tiling lattice;
3. $L$ is expanded from the crossingless unknot by a finite sequence of moves I and II;
4. $L$ may be converted to the $k$-component unlink by a sequence of $(k-1)$ band moves and finitely many Conway mutations for some $k \in \mathbb{N}$. 
Proof sketch

1. $\Sigma_2(S^3,L)$ bounds a QHB;
2. $\Lambda_b$ is a 2-cube tiling lattice;
3. $L$ is expanded from the crossingless unknot $\emptyset$ by a finite sequence of moves I and II;
4. $L$ may be converted to the $k$-component unlink by a sequence of $(k-1)$ band moves and finitely many Conway mutations for some $k \in \mathbb{N}$.

$\Rightarrow$ : Take double branched cover
Thanks!