Composition series of a class of induced representations

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$$\delta_1 \times \cdots \times \delta_k \rtimes \tau = \mathrm{Ind}_{M_s}^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$$

normalized parabolic induction.

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$e([\nu^x\rho, \nu^y\rho]) = e(\delta([\nu^x\rho, \nu^y\rho]) = \frac{x+y}{2}$.

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$\delta(\Delta) \times \delta(\Delta') \cong \delta(\Delta') \times \delta(\Delta)$ irreducible, $\Delta' \subseteq \Delta$ segments

Geometric lemma

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} \text{s.s.}(r_{(k)}(\sigma))$$

$$\mu^*(\delta([\nu^x\rho, \nu^y\rho]) \rtimes \sigma) = \sum_{\delta'\otimes\sigma' \leq \mu^*(\sigma)} \sum_{i=0}^{y-x+1} \sum_{j=0}^{i} \delta([\nu^{i-y}\rho, \nu^{-x}\rho]) \times \delta([\nu^{y+1-j}\rho, \nu^y\rho]) \times \delta' \otimes \delta([\nu^{y+1-i}\rho, \nu^{y-j}\rho]) \rtimes \sigma'$$

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We have $\delta([\nu^x\rho, \nu^y\rho]) \rtimes \sigma = \delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \sigma.$

Langlands classification

$$\sigma \quad \longleftrightarrow \quad \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \tau \twoheadrightarrow \sigma$$

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$\sigma = \mathrm{Lang}(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \tau)$ is of multiplicity 1 in the induced representation.

Mœglin-Tadić classification of discrete series

$$\sigma \quad \longleftrightarrow \quad (\mathrm{Jord}, \sigma_{cusp}, \epsilon)$$

$$\mathrm{Jord} = \{(a,\rho) \mid a \in \mathbb{N},\ \delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$$
$$\text{is irreducible and reduces for some larger } a \text{ of the same parity}\}$$

$\sigma \hookrightarrow \pi \rtimes \sigma_{cusp}$, for some $\pi \in \mathrm{Irr\ GL}$ and σ_{cusp} is irreducible cuspidal,

$\epsilon :\ \subseteq \mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord}) \to \{\pm 1\}$

Assume that $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces.

$$\epsilon((a,\rho),(a',\rho)) = \epsilon(a,\rho)\epsilon(a',\rho)^{-1} = \epsilon(a,\rho)\cdot\epsilon(a',\rho)^{-1}$$

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$\mathrm{Jord}_\rho = \{a \mid (a,\rho) \in \mathrm{Jord}\}$
$(a,\rho) \in \mathrm{Jord}$
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$$\epsilon(a,\rho)\epsilon(a_-,\rho)^{-1} = 1 \Leftrightarrow \text{there exists } \pi' \in \mathrm{Irr}(G_{n_{\pi'}}) \text{ such that } \sigma \hookrightarrow \delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'$$

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For $a = \min(\mathrm{Jord}_\rho)$ we have

$$\epsilon(a,\rho) = 1 \Leftrightarrow \text{there exists } \pi'' \in \mathrm{Irr}(G_{n_{\pi''}}) \text{ such that } \sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi''.$$

Goal: determine composition series of

$$\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma$$

where
$\frac{1}{2} \leq a < b < c \in \frac{1}{2} + \mathbb{Z}$
ρ is irreducible cuspidal unitarizable representation of $GL(m_\rho, F)$
σ is irreducible cuspidal representation of G_n
$\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces.

Basic reducibilities

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma = \sigma_1 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma).$$

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$$\begin{aligned}\mathrm{Jord}_\rho(\sigma_2) = \mathrm{Jord}_\rho(\sigma_3) = \{(2b+1,\rho), (2c+1,\rho)\},\\ \epsilon_{\sigma_2}(2b+1,\rho) = \epsilon_{\sigma_2}(2c+1,\rho) = 1,\\ \epsilon_{\sigma_3}(2b+1,\rho) = \epsilon_{\sigma_3}(2c+1,\rho) = -1.\end{aligned}$$

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$$\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1 = \sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1).$$

$$\begin{aligned} \mathrm{Jord}_\rho(\sigma_4) = \mathrm{Jord}_\rho(\sigma_5) = \{(2a+1, \rho), (2b+1, \rho), (2c+1, \rho)\}, \\ \epsilon_{\sigma_4}(2a+1, \rho) = \epsilon_{\sigma_4}(2b+1, \rho) = \epsilon_{\sigma_4}(2c+1, \rho) = 1, \\ \epsilon_{\sigma_5}(2a+1, \rho) = 1, \epsilon_{\sigma_5}(2b+1, \rho) = \epsilon_{\sigma_5}(2c+1, \rho) = -1. \end{aligned}$$

We have:

representation $\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma$ has two irreducible representations, σ_4 i σ_5, they appear with multiplicity one in the induced representation.

Also

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 = \sigma_4 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2),$$
$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 = \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3).$$

Teorem

$$\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma=$$
$$\sigma_4+\sigma_5+\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma_2)+\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma_3)$$
$$+\mathrm{Lang}(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_1)+\mathrm{Lang}(\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma)$$

We have a filtration

$$\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma_2)\oplus\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma_3)\oplus$$
$$\mathrm{Lang}(\delta([\nu^{-b}\rho,\nu^{c}\rho])\rtimes\sigma_1)$$
$$\hookrightarrow(\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma)/(\sigma_4\oplus\sigma_5).$$

Proof. Assume $-b+c \geq \frac{1}{2}+a$. Observe intertwinings

$$
\begin{aligned}
&\delta([\nu^{-b}\rho,\nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma \\
&\qquad \cong \delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \times \delta([\nu^{-b}\rho,\nu^{c}\rho]) \rtimes \sigma \\
&\qquad \to \delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \times \delta([\nu^{-c}\rho,\nu^{b}\rho]) \rtimes \sigma \\
&\qquad \cong \delta([\nu^{-c}\rho,\nu^{b}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma \\
&\qquad \to \delta([\nu^{-c}\rho,\nu^{b}\rho]) \times \delta([\nu^{-a}\rho,\nu^{-\frac{1}{2}}\rho]) \rtimes \sigma.
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\end{aligned}$$

The kernel of the second map is
$\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma_2 + \delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma_3$, that is

$$\sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma_2) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma_3).$$

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$$\sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma_2) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho]) \rtimes \sigma_3).$$

The kernel of the last map is $\delta([\nu^{-c}\rho,\nu^{b}\rho]) \rtimes \sigma_1 = \delta([\nu^{-b}\rho,\nu^{c}\rho]) \rtimes \sigma_1$, that is

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The image of the composition is $\mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)$.

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Now we have

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma$$

$$\mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1) \hookrightarrow (\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5)$$

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Also

$$\begin{aligned}\sigma_4 \oplus \sigma_5 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 \\ &\hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma\end{aligned}$$

$$\begin{aligned}&\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2) \oplus \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3) \hookrightarrow \\ &\qquad (\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5)\end{aligned}$$

The second formula follows.