Symmetries of bi-Cayley graphs

Jin-Xin Zhou

Beijing Jiaotong University, Beijing 100044, P.R. China

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Outline

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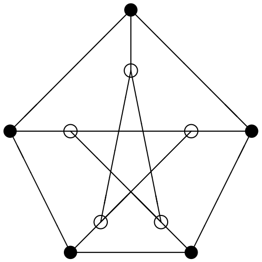

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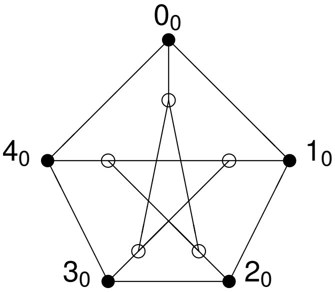

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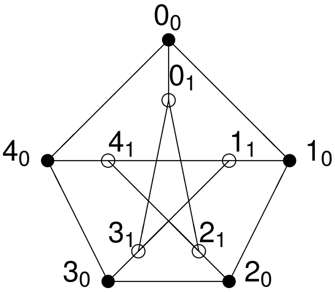

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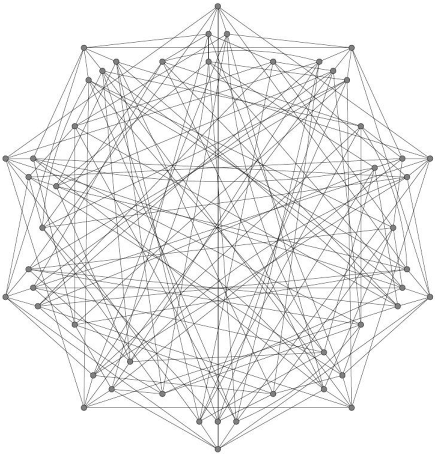

Figure 2: Hoffman-Singleton graph.

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How to decide if a given graph is a bi-Cayley graph?

Automorphisms of graphs

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All automorphisms of Γ form a permutation group on V, called **the full automorphism group** of Γ, denoted by $\mathrm{Aut}(\Gamma)$.

Bi-regular representation

Let Γ =BiCay(H, R, L, S) be a bi-Cayley graph

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For each $g \in H$, let

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Then

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is a group of automorphisms of BiCay(H, R, L, S) acting semiregularly on its vertices with two orbits.

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Bi-Cayley graph test

A graph Γ is a *bi-Cayley graph* over a group H if and only if $\mathrm{Aut}(\Gamma)$ contains a semiregular subgroup isomorphic to H having 2 orbits on $V(\Gamma)$.

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A graph Γ is a *Cayley graph* on a group G if and only if $\mathrm{Aut}(\Gamma)$ has a subgroup isomorphic to G and acting regularly on the vertices of Γ.

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Godsil Theorem (Godsil, Combinatorica, 1981)

The normalizer of $R(G)$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is $R(G) \rtimes \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing the set S set-wise.

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A lot of people have been working on this area: L. Babai, S.F. Du, X.G. Fang, Y.-Q. Feng, C.D. Godsi, L.A. Nowitz, M.E. Watkins, C.H. Li, C.E. Praeger, M.Y. Xu, ...

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The first natural step is to determine the normalizer of the group $\mathcal{R}(H)$ in $\mathrm{Aut}(\Gamma)$, where Γ is a bi-Cayley graph of the group H.

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We will give a solution of this problem.

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Set

$$I = \{\delta_{\alpha,x,y} \mid \alpha \in \mathrm{Aut}(H) \ s.t. \ \mathbf{R}^\alpha = \mathbf{x}^{-1}\mathbf{L}\mathbf{x}, \ \mathbf{L}^\alpha = \mathbf{y}^{-1}\mathbf{R}\mathbf{y}, \ \mathbf{S}^\alpha = \mathbf{y}^{-1}\mathbf{S}^{-1}\mathbf{x}\},$$
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Theorem 1 (Z. & Feng, JCTB, 2016)

Let Γ =BiCay(H, R, L, S) be a connected bi-Cayley graph over the group H. Then

- $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H)) = \mathcal{R}(H) \rtimes F$ if $I = \emptyset$,
- $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H)) = \mathcal{R}(H)\langle F, \delta_{\alpha,x,y}\rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$.

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If $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H)) = \mathrm{Aut}(\Gamma)$, then Γ is called a **normal bi-Cayley graph** over H.

Petersen graph has a solvable VT group of automorphisms

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Let $H = \mathbb{Z}_5$. Then $\mathcal{R}(1) = (0_0, 1_0, 2_0, 3_0, 4_0)(0_1, 1_1, 2_1, 3_1, 4_1)$, and so $\mathcal{R}(H) = \langle \mathcal{R}(1) \rangle$.

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Let $\alpha \in \mathrm{Aut}(H)$ be such that $\alpha(1) = 2$. Then α swaps $\{2,3\}$ and $\{1,4\}$. So $\delta_{\alpha,1,1}$ is an automorphism of $P(5,2)$ which interchanges H_0 and H_1 and normalizes $\mathcal{R}(H)$.

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So $\mathcal{R}(H) \rtimes \langle \delta_{\alpha,1,1} \rangle \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ is vertex transitive on $P(5,2)$.

Basic properties of $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$

Let $X = N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$. Note that

$$X_{1_0 1_1} = \langle \sigma_{\alpha,1} \mid \alpha \in \mathrm{Aut}(H), S^\alpha = S \rangle.$$

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Lemma 2

Let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph over a group H, with $1_H \in S$. Let $X = N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$. Then X_v acts faithfully on the neighborhood of v.

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Γ is s-arc-transitive: $\mathrm{Aut}(\Gamma)$ is transitive on s-arcs.

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Theorem 3 (Conder, Z., Feng & Zhang, JCTB, 2020)

Let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph over a group H, with $1_H \in S$. Then $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$ acts transitively on the 2-arcs of Γ if and only if the following three conditions hold :

(a) there exists an automorphism α of H such that $S^\alpha = S^{-1}$,

(b) the setwise stabilizer of S in $\mathrm{Aut}(H)$ is transitive on $S \setminus \{1_H\}$, and

(c) there exists $s \in S \setminus \{1_H\}$ and an automorphism β of H such that $S^\beta = s^{-1}S$.

Furthermore, $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$ is not transitive on the 3-arcs of Γ.

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Theorem 3 (Conder, Z., Feng & Zhang, JCTB, 2020)

Let $\Gamma = \mathrm{BiCay}(H, \emptyset, \emptyset, S)$ be a connected bi-Cayley graph over a group H, with $1_H \in S$. Then $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$ acts transitively on the 2-arcs of Γ if and only if the following three conditions hold :

(a) there exists an automorphism α of H such that $S^\alpha = S^{-1}$,

(b) the setwise stabilizer of S in $\mathrm{Aut}(H)$ is transitive on $S \setminus \{1_H\}$, and

(c) there exists $s \in S \setminus \{1_H\}$ and an automorphism β of H such that $S^\beta = s^{-1}S$.

Furthermore, $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$ is not transitive on the 3-arcs of Γ.

So, a normal bi-Cayley graph is at most 2-arc-transitive.

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Problem A (C.H. Li, Proc. of AMC, 2005)

- Do there exist 3-arc-transitive bi-normal Cayley graphs?
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- Do there exist 3-arc-transitive bi-normal Cayley graphs?
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Clearly, every **bi-normal Cayley graph** is a **normal bi-Cayley graph**.

A question

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Problem A

What can we say about non-bipartite s-arc-transitive graphs which have a vertex-transitive solvable group of automorphisms?

Li and Xia (Mem. Amer. Math. Soc. 2021+) have made a significant progress towards this problem.

Let Γ be a (G, s)-arc-transitive graph with $G \leq \mathrm{Aut}(\Gamma)$ and $s \geq 2$, and let N be a normal subgroup of G.

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Theorem 4 (Li & Xia, Mem. Amer. Math. Soc. to appear)

A connected non-bipartite 3-arc-transitive Cayley graph on a solvable group of valency at least three is a normal cover of the Petersen graph or the Hoffman-Singleton graph.

No non-bipartite 3-arc-transitive solvable Cayley graph

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Recall that both Petersen graph and Hoffman-Singleton graph are bi-Cayley graphs.

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Using this fact, together with some other methods, we prove the following:

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Theorem 5 (Z. JCTB, 2021)

Every connected non-bipartite Cayley graph on a solvable group of valency at least three is at most 2-arc-transitive.

When is a bi-Cayley graph normal?

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Let p be a prime and let Γ be a connected bi-Cayley graph over a non-abelian metacyclic p-group H.

If Γ is cubic edge-transitive, then $p = 3$ and Γ is either the Gray graph or a normal bi-Cayley graph over H (Qin., Z., EleJC, 2018).

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If $p > 3$ and Γ is tetravalent, vertex- and edge-transitive, then Γ is a normal bi-Cayley graph over H (Conder, Z., Feng, Zhang, JCTB, 2020).

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If Γ is bipartite and $\mathcal{R}(H)$ is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$, then Γ is a normal bi-Cayley graph over H (Feng, Wang, ARS Math. Contemp. 2019).

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Recently, Li, Zhang, Z. investigated the bi-primitive s-arc-transitive bi-partite bi-Cayley graphs, and obtain the following:

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Theorem 6 (Li, Zhang, Z., 2021+)

Let Γ be a bi-primitive s-arc-transitive bi-partite bi-Cayley graph over a group H with $s \geq 2$. Then either $\mathrm{Aut}(\Gamma)^+$ is of PA-type, or one of the following holds:

(1) Γ is a normal bi-Cayley graph;

(2) $\Gamma \cong K_{n,n}$;

(3) Γ is the standard double cover of K_n or a vertex-primitive s-arc-transitive graph [1];

(4) $\Gamma \cong HP(n-1,q)$, $\overline{HP(n-1,q)}$ ($n \geq 3$), $G(22,5)$ or $B'(H(11))$ (see [2]);

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1. J.J. Li, J. Yang, W.Y. Zhu, Vertex primitive *s*-transitive Cayley graphs, Discrete Math. 343 (2020) 1–6.

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1. J.J. Li, J. Yang, W.Y. Zhu, Vertex primitive s-transitive Cayley graphs, Discrete Math. 343 (2020) 1–6.

2. Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, J. Comb. Theory, Ser. B 42 (1987) 196–211.

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Follow on this, we say that $\Gamma = \mathrm{BiCay}(H, R, L, S)$ is **normal-edge-transitive** if $N_{\mathrm{Aut}(\Gamma)}(\mathcal{R}(H))$ is transitive on the edges of Γ.

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Theorem 7 [Z. & Feng, 2016]

Let Γ be a connected cubic edge-transitive bi-Cayley graph $\mathrm{BiCay}(H, \emptyset, \emptyset, S)$ over a 2-group H. Then Γ is normal if and only if Γ is normal-edge-transitive.

A class of cubic edge-transitive graphs

Let $n \geq 2$ be a positive integer, and let $\mathcal{G}(n) = \langle a, b, c, d, e, x, y \rangle$ with the following relations:

$$\begin{aligned} a^{2^n} = b^{2^n} = c^4 = d^2 = e^2 = x^2 = y^2 = 1, \\ c = [a, b], d = [a, c], e = [b, c], x = [c, d], y = [c, e], \\ [e, d] = [x, a] = [x, b] = [y, a] = [y, b] = 1, \\ d^a = yd, e^a = c^2 e, d^b = xyc^2 d, e^b = xye. \end{aligned} \tag{1}$$

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Theorem 8 (Z. & Feng, JCTB, 2016)

Let $\Gamma = \mathrm{BiCay}(\mathcal{G}(n), \emptyset, \emptyset, \{1, a, b\})$. Then Γ is a connected cubic 1-arc-regular normal bi-Cayley graph over $\mathcal{G}(n)$. Furthermore, there exists $\delta \in \mathrm{Aut}(\mathcal{G}(n))$ such that $a^\delta = b^{-1}$ and $b^\delta = a^{-1}$, and $\Gamma \cong \mathrm{Cay}(G, T)$ is a non-normal Cayley graph on G, where $G = \mathcal{G}(n) \rtimes \langle \delta \rangle$ and $T = \{\delta, \delta a, \delta b\}$.

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Problem B (Godsil, EJC, 1983)

Suppose that $\Gamma = \mathrm{Cay}(G, S)$ is a cubic Cayley graph for the 2-group G and let $A = \mathrm{Aut}(\Gamma)$. Is it true that if $|A_1| \neq 1$ then $\mathrm{Aut}(G, S)$ is non-trivial?

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The answer is 'yes' if $2 \mid |A_1|$.

However, if $2 \nmid |A_1|$, then there exist 1-arc-regular normal bi-Cayley graphs over 2-groups which are non-normal Cayley graphs.

Normal-edge-transitive bi-dihedrants

Let n and k be integers with $n \geq 5$ and $k \geq 2$, such that there exists an element λ of order $2k$ in $\mathbb{Z}_n^*$ such that

$$1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2(k-2)} + \lambda^{2(k-1)} \equiv 0 \mod n.$$

Now let $H = D_{2n} = \langle\, a, b \mid a^n = b^2 = (ab)^2 = 1 \,\rangle$, and for each $i \in \mathbb{Z}_k$, let

$$c_i = 1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2(i-1)} + \lambda^{2i},$$
$$d_i = \lambda c_i = \lambda + \lambda^3 + \lambda^5 + \cdots + \lambda^{2i-1} + \lambda^{2i+1},$$

and then define $\Gamma(n, \lambda, 2k)$ as the $2k$-valent bi-Cayley graph $\mathrm{BiCay}(H, \emptyset, \emptyset, S)$ over H, where

$$S = S(n, \lambda, 2k) = \{a^{c_i} : i \in \mathbb{Z}_k\} \cup \{ba^{d_i} : i \in \mathbb{Z}_k\}.$$

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Determine which of the graphs $\Gamma(n, \lambda, 2k)$ are semisymmetric.

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Proposition 9(Conder, Z., Feng & Zhang, 2020)

If k is even and $\lambda^k \equiv -1 \bmod n$, then $\Gamma(n, \lambda, 2k)$ is arc-transitive.

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The graph $\Gamma(n, \lambda, 2k)$ is semisymmetric whenever $k = 3$, and moreover, if $k = 3$ and $\lambda^3 \not\equiv -1 \bmod n$, then $\Gamma(n, \lambda, 2k)$ is edge-regular, with cyclic vertex-stabilizer.

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Conjecture D(Conder, Z., Feng & Zhang, 2020)

$\Gamma(n, \lambda, 2k)$ is arc-transitive if and only if k is even and $\lambda^k \equiv -1$ mod n.

Thanks!