Inverses of $k$-Toeplitz matrices for resonator arrays with multiple receivers

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Wireless power transfer (WPT) is used to transfer energy while avoiding electrical contact.

- Useful for safety measures and in rough environments: water, dust, dirt.
- E.g.: electrical vehicle charging, biomedical devices.
Electromagnetic induction: power is transferred between coils through a magnetic field (by Ampere’s law)

Circuit representation: mutual inductance $M$
WPT over long distances

WPT over longer distances can be achieved via an array of parallel resonators

The receiver absorbing the power is placed over some resonator
WPT over long distances

- The array could transmit power to several receivers.
- Cases with more than one receiver were difficult to analyze until now.
- We study the case with receivers placed periodically over the array.
Circuit analysis

- We have to analyze an equivalent circuit with resistors, capacitors and inductors, together with their mutual inductances, which is periodic.

- Given the source voltage and the component parameters, we want to find the currents through each component, to compute power transference and efficiency.
The Fourier transform allows to analyze the circuit with linear algebra.

For a fixed frequency $\omega$ each component $X$ behaves like a resistor in the frequency domain:

$$V_X = Z_X I_X$$

$Z_X$ is called the impedance of the component $X$.

<table>
<thead>
<tr>
<th>Component</th>
<th>Time domain</th>
<th>Impedance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>$V = RI$</td>
<td>$Z_R = R$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$V = L \frac{dI}{dt}$</td>
<td>$Z_L = i\omega L$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$V = \frac{1}{C} \int I dt$</td>
<td>$Z_C = \frac{1}{i\omega C}$</td>
</tr>
</tbody>
</table>
Circuit analysis

- The **current-voltage relations** of the circuit are expressed in the matrix equation

  \[ AI = V, \]

  where \( A \) the **impedance matrix**, \( V \) the vector of voltage sources, \( I \) the vector of unknown currents

- To fully characterize the system **it is enough to invert this matrix**:

  \[ I = A^{-1}V \]
In our case we get a tridiagonal $k$-Toeplitz matrix with constant and equal upper and lower diagonals:

$$A = \begin{pmatrix}
Z & i\omega M & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\
i\omega M & Z & i\omega M & 0 & \ldots & \ldots & \ldots & 0 \\
0 & i\omega M & Z + Z_d & i\omega M & 0 & \ldots & \ldots & 0 \\
0 & 0 & i\omega M & Z & i\omega M & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & i\omega M \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & Z + Z_T
\end{pmatrix}$$
We define $M_n(a_1, \ldots, a_k, b) \in M_n(\mathbb{C})$ as

$$
\begin{pmatrix}
    a_1 & b & 0 & \cdots & \cdots & 0 \\
    b & \ddots & \ddots & \ddots & \ddots & \\
    0 & \ddots & a_k & b & \ddots & \\
    \vdots & \ddots & b & a_1 & \ddots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & b \\
    0 & \cdots & \cdots & \cdots & b & a_m
\end{pmatrix}
$$

where $m := n \, (\text{mod} \, k)$ and $a_0 := a_k$. 
Computing the inverse

We compute the inverse of $A := M_n(a_1, \ldots, a_k, b)$ through the adjugate matrix

$$A^{-1} = \frac{\text{adj}(A)}{\det A}$$

$\text{adj}(A) = (C_{ij})_{i,j=1}^n$ is formed by the cofactors

$$C_{ij} := (-1)^{i+j} \det(A_{ij})$$

($A$ is symmetric)

$A_{ij}$ the submatrix formed by removing $i$th row and $j$th column
Computing the inverse: cofactors

The submatrix $A_{ij}$ is upper block-triangular with three diagonal blocks:

$$A = \begin{pmatrix}
A_1 & b \\
 b & a_i & b \\
 b & a_{i+1} & b \\
 b & \ddots & b \\
 b & a_j & b \\
 b & A_2
\end{pmatrix}$$
The submatrix $A_{ij}$ is upper block-triangular with three diagonal blocks:

$$A_{ij} = \begin{pmatrix} A_1 & b & & & \\ b & a_{i+1} & & & \\ & b & \ddots & & \\ & & & b & \\ & & & & A_2 \end{pmatrix}$$
Computing the inverse: cofactors

The submatrix $A_{ij}$ is upper block-triangular with three diagonal blocks:

$$A_{ij} = \begin{pmatrix}
A_1 & b \\
 & \begin{pmatrix}
 b & a_{i+1} \\
 & \begin{pmatrix}
 b & \cdots \\
 & b & a_{j-1} \\
 & A_2
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}$$

- $A_1 = M_{i-1}(a_1, \ldots, a_k, b)$
- $A_2 = M_{n-j}(\sigma_j(a_1, \ldots, a_k), b)$ with $\sigma_j$ the $j$th cyclic permutation to the left, $\sigma_0 = \text{id}$
- The middle block $B$ has determinant $b^{j-i}$
Computing the inverse

- We need $C_{ij} = (-1)^{i+j} \det(A_{ij})$
- The determinant of a block-triangular matrix is the product of the determinants of its diagonal blocks
  \[
  \det(A_{ij}) = \det(A_1) \det(B) \det(A_2)
  \]
- Denote $D_n(a_1, \ldots, a_k, b) := \det(M_n(a_1, \ldots, a_k, b))$
- The $(i, j)$th element of the inverse $A^{-1}$ is
  \[
  (-b)^{j-1} \frac{D_{i-1}(a_1, \ldots, a_k, b) D_{n-j}(\sigma_j(a_1, \ldots, a_k), b)}{D_n(a_1, \ldots, a_k, b)}
  \]
Computing the inverse: determinant

- Denote \( D(n) := D_n(a_1, \ldots, a_k, b) \)

\[
D(n) = \begin{vmatrix}
  a_1 & b & 0 & \ldots & \ldots & 0 \\
  b & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & \ddots & a_k & b & \ddots & \vdots \\
  \vdots & \ddots & b & a_1 & \ddots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & b \\
  0 & \ldots & \ldots & 0 & b & a_m
\end{vmatrix}
\]

- By **Laplace expansions** along last column and last row

\[
D(n) = a_mD(n-1) - b^2D(n-2)
\]
We find $D(n)$ as the solution of a system of $k$ linear recurrence equations. In matrix form:

\[
\begin{pmatrix}
D(kn) \\
D(kn - 1)
\end{pmatrix} = 
\begin{pmatrix}
 a_k & -b^2 \\
 1 & 0
\end{pmatrix} 
\begin{pmatrix}
D(kn - 1) \\
D(kn - 2)
\end{pmatrix},
\]

\[
\begin{pmatrix}
D(kn - 1) \\
D(kn - 2)
\end{pmatrix} = 
\begin{pmatrix}
 a_{k-1} & -b^2 \\
 1 & 0
\end{pmatrix} 
\begin{pmatrix}
D(kn - 2) \\
D(kn - 3)
\end{pmatrix},
\]

\[\vdots\]

\[
\begin{pmatrix}
D(k(n - 1) + 1) \\
D(k(n - 1))
\end{pmatrix} = 
\begin{pmatrix}
 a_1 & -b^2 \\
 1 & 0
\end{pmatrix} 
\begin{pmatrix}
D(k(n - 1)) \\
D(k(n - 1) - 1)
\end{pmatrix}.
\]
Computing the inverse: determinant

So

\[
\begin{pmatrix} D(kn) \\ D(kn - 1) \end{pmatrix} = C^{n-1} \begin{pmatrix} D(k) \\ D(k - 1) \end{pmatrix}
\]

with

\[
C := \begin{pmatrix} a_k & -b^2 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -b^2 \\ 1 & 0 \end{pmatrix}
\]

The other \( k - 2 \) determinants of the form \( D(kn - r) \) are computed from the previous ones.

\( C^{n-1} \) is computed by diagonalization.
Example: For $k = 3$:

- Put $a^3 := a_1 a_2 a_3$, $s := a_1 + a_2 + a_3$, $d(a_1, a_2) := a_1 a_2 - b^2$
- We get

$$\begin{pmatrix} D(3n) \\ D(3n - 1) \end{pmatrix} = C^{n-1} \begin{pmatrix} a^3 + (a_2 - s)b^2 \\ d \end{pmatrix}$$

- $C$ has characteristic polynomial $X^2 + (sb^2 - a^3)X + b^6$ with eigenvalues $r_{1,2}$. Computing $C^{n-1}$ we find

$$D(3n) = \frac{1}{r_1 - r_2} ((r_1 + a_2 b^2)r_1^n - (r_2 + a_2 b^2)r_2^n),$$
$$D(3n - 1) = \frac{d}{r_1 - r_2} (r_1^n - r_2^n).$$

- $D(3n - 2) = a_1 D(3(n - 1)) - b^2 D(3(n - 1) - 1)$, so

$$D(3n - 2) = \frac{1}{r_1 - r_2} ((a_1 r_1 + b^4)r_1^{n-1} - (a_1 r_2 + b^4)r_2^{n-1}).$$
We get **rational formulas** for the currents, powers, and efficiencies of the WPT system.

**Example:** Efficiency for \( N = 3n \), a receptor in each multiple of 3:

\[
\eta_{3n} = \frac{R_d (R^2 + (\omega_0 M)^2)}{(r_3^n - r_4^n)(t_3 r_3^{n-1} - t_4 r_4^{n-1})} \sum_{j=1}^{n-1} (\omega_0 M)^{6j-2} (r_3^{n-j} - r_4^{n-j})^2
\]

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**Brox, Alberto.** *Inverses of \( k \)-Toeplitz matrices with applications to resonator arrays with multiple receivers.* Applied Mathematics and Computation