The topological conjugacy criterion for surface Morse-Smale flows with a finite number of moduli

Vladislav E. Kruglov, HSE University, Nižnij Novgorod, Russia

8ECM 2021, Portorož

25.06.2021
The results were obtained in collaboration with Olga V. Pochinka
Topological conjugacy and equivalence

• Two flows $f^t, f'^t : M \to M$ on a manifold $M$ are called topologically equivalent if there exists a homeomorphism $h : M \to M$ sending trajectories of $f^t$ into trajectories of $f'^t$ preserving orientations of the trajectories.

• Two flows are called topologically conjugate if $h \circ f^t = f'^t \circ h$, it means that $h$ sends trajectories into trajectories preserving not only directions but in addition the time of moving.

• To find an invariant showing the class of topological equivalence or topological conjugacy of flows in some class means to get a topological classification for the class.
The Morse-Smale flows

A flow on a surface is called Morse-Smale if its non-wandering set consists of a finite number of hyperbolic fixed points and finite number of hyperbolic limit cycles, besides, there is no trajectories connecting saddle points.

The most important topological invariants for Morse-Smale flows are the Leontovich-Maier’s scheme [4, 5], the Peixoto’s directed graph [6] and the Oshemkov-Sharko’s molecule [7].
The moduli of stability

- A separatrix connecting saddle points gives infinitely many conjugacy classes in one equivalence class, described by a modulus \( \frac{\lambda}{\mu} \) called the *modulus of stability* (Palis, 1978).

\[ \lambda \quad \mu \]

- For surface gradient-like flows classes of topological equivalence and topological conjugacy on surfaces coincide (Kruglov, [2]).

- Any limit cycle obviously generates a *modulus* equal to its period.
The problems solved in the work

• The criterion of the moduli finiteness for the surface Morse-Smale flows;
• Topological classification in sense of conjugacy for surface Morse-Smale flows with a finite number of moduli.
Fixed points

The hyperbolicity of fixed points leads to the following types of fixed points: a sink, a saddle and a source. A flow near a fixed point is topologically conjugate with a linear flow with a sink, saddle or source respectively (Palis, de Melo [9], Robinson [10], Kruglov [2]).
Limit cycles

The hyperbolicity of limit cycles leads to the fact that limit cycles may be only stable or unstable. The neighbourhood of a limit cycle is an annulus or a Möbius band.
We define a flow $A^t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $A^t(x, y) = (x, y + t)$. For $\mu \in \{-1, 1\}$, $\lambda \in \{0, 1\}$ and $T > 0$ let us consider a homeomorphism $g_{\mu, \lambda, T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the formula

$$g_{\mu, \lambda, T}(x, y) = (\mu \cdot 2^{(-1)^{\lambda+1}}, y - T)$$

and the group $G_{\mu, \lambda, T} = \{g_{\mu, \lambda, T}^n, n \in \mathbb{Z}\}$. Denote by $\Pi_{\mu, \lambda, T}$ a space orbit of the action of the group $G_{\mu, \lambda, T}$ on $\mathbb{R}^2$ and by $q_{\mu, \lambda, T} : \mathbb{R}^2 \rightarrow \Pi_{\mu, \lambda, T}$ the natural projection. Then $\Pi_{\mu, \lambda, T}$ is a cylinder for $\mu = 1$ and a Möbius band for $\mu = -1$; the flow $A^t$ induces by $q_{\mu, \lambda, T}$ the flow $a_{\mu, \lambda, T}^t$ on $\Pi_{\mu, \lambda, T}$ with unique stable limit cycle $c_{\mu, \lambda, T} = q_{\mu, \lambda, T}(Oy)$ of the period $T$ for $\lambda = 0$, and the flow $a_{\mu, \lambda, T}^t$ on $\Pi_{\mu, \lambda, T}$ with unique unstable limit cycle $c_{\mu, \lambda, T} = q_{\mu, \lambda, T}(Oy)$ of the period $T$ for $\lambda = 1$. 
Proposition (Irwin [1])

For every hyperbolic limit cycle \( c_i \) of a flow \( \phi^t : S \to S \) there are numbers \( \mu_i \in \{-1, 1\} \), \( \lambda_i \in \{0, 1\} \), \( T_i > 0 \) and a neighbourhood \( U_i \) such that \( \phi^t|_{U_i} \) is topologically conjugate to the flow \( a^t_{\mu_i,\lambda_i,T_i} \).

The limit cycle \( c_i \) is called a \textit{stable}, \textit{an unstable} for \( \lambda_i = 0, 1 \) respectively.
The unique foliation near limit cycle

Let $K_i = W^u_{\Omega_i}$ for an unstable cycle $\Omega_i$ and $K_i = W^s_{\Omega_i}$ for a stable cycle $\Omega_i$, respectively.

**Proposition (Kruglov, Pochinka, Talanova, [3])**

There is a unique one-dimensional foliation $\Xi_i$ in $K_i$ whose leaves $\xi_i$ are cross-sections for trajectories of flow $\phi^t|_{K_i}$ and

$$\phi^{T_i}(z) \in \xi_i, \, \phi^t(z) \notin \xi_i \text{ for } 0 < t < T_i, \text{ if } z \in \xi_i.$$
Recall that a modulus of topological conjugacy is an analytical parameter describing infinite many conjugacy classes in the equivalence class.

The first main result of the report is the following.

**Theorem**

A Morse-Smale surface flow has a finite number of moduli iff it has no a trajectory going from one limit cycle to another.

Let $G$ be the class of Morse-Smale flow with a finite number of moduli, and let $\phi^t \in G$. 

Cutting set and cutting circles. An elementary region

Let \( \mathcal{R} = \bigcup_{c \in \Omega_{\phi^t}} R_c \) be the union of the boundary circles of cycles’ es neighbourhoods. We call \( \mathcal{R} \) a cutting set and the connected components of \( \mathcal{R} \) cutting circles. Let \( \hat{S} = S \setminus \mathcal{R} \). We call an elementary region a connected component of the set \( \hat{S} \). The elementary regions, obviously, can be of the following pairwise disjoint types with respect to information about basic sets of \( \phi^t \) in the regions:

1) a region of the type \( \mathcal{L} \) contains exactly one limit cycle;

2) a region of the type \( \mathcal{A} \) contains exactly one source or exactly one sink;

3) a region of the type \( \mathcal{M} \) contains at least one saddle point;
The directed graph of a flow

Definition

A directed graph $\Upsilon_{\phi^t}$ is said to be a graph of the flow $\phi^t \in G$ if

1. the vertices of $\Upsilon_{\phi^t}$ bijectively correspond to the elementary regions of $\phi^t$;

2. every directed edge of $\Upsilon_{\phi^t}$, which joins a vertex $a$ with a vertex $b$, corresponds to the cutting circle $R$, which is a common boundary of the regions $A$ and $B$ corresponding to $a$ and $b$, such that any trajectory of $\phi^t$ passing $R$ goes from $A$ to $B$ by increasing the time.
Properties of the directed graph

We will call a $\mathcal{L}$-, $\mathcal{A}$-, or $\mathcal{M}$-vertex a vertex of $\Upsilon_{\phi^t}$, which corresponds to a $\mathcal{L}$-, $\mathcal{A}$-, or $\mathcal{M}$-region accordingly.

**Proposition**

Let $\Upsilon_{\phi^t}$ be the directed graph of a flow $\phi^t \in G$, then:

1) every $\mathcal{M}$-vertex can be connected only with $\mathcal{L}$-vertices, furthermore, with every vertex by a single edge;

2) every $\mathcal{A}$-vertex can be connected only with a $\mathcal{L}$-vertex, furthermore, by a single edge;

3) every $\mathcal{L}$-vertex has degree (the number of incident edges) 1 or 2, and if its degree is 2, then both edges either enter the vertex or exit.
Equipping of the graph $\Upsilon_{\phi^t}$

The flows in $\mathcal{A}$-regions can belong to only the two conjugacy classes: a source pool and a sink pool, which we can distinguish by directions of edges incident to $\mathcal{A}$-vertices.

The flows in $\mathcal{L}$-regions can belong to only the four equivalence classes:
– an annulus with a stable limit cycle;
– an annulus with an unstable one;
– the Möbius band with a stable one;
– the Möbius band with an unstable one.

But every equivalence class consists of infinitely many conjugacy classes depending on a period of limit cycles. So, let us equip each $\mathcal{L}$-vertex with a cycle modulus, i.e. the period.
Equipping of $\mathcal{M}$-vertex. Constructing a surface $M$ and a gradient-like flow on it

Consider an $\mathcal{M}$-region. It can be
- a 2-manifold with a boundary (with “holes”);
- a closed surface;

Attach a union $D$ of 2-disk to each boundary component of $\mathcal{M}$ to get a closed surface $M$.

Let $f^t : M \to M$ be the flow such that $f^t|_{\mathcal{M}} = \phi^t|_{\mathcal{M}}$ and that $\Omega_{f^t}$ has exactly one sink or one source in each connected component of $D$.
Equipping of $\mathcal{M}$-vertex. A cell

Let $\Omega_{f^t}^0$, $\Omega_{f^t}^1$, $\Omega_{f^t}^2$ be the sets of all sources, saddle points and sinks of $f^t$ accordingly. By the definition of the region $\mathcal{M}$ the flow $f^t$ has at least one saddle point. Let

$$\tilde{M} = M \setminus (\Omega_{f^t}^0 \cup W^s_{\Omega_{f^t}^1} \cup W^u_{\Omega_{f^t}^1} \cup \Omega_{f^t}^2).$$

A connected component of $\tilde{M}$ is called a cell.

**Proposition (Peixoto [7])**

Every cell $J$ of the flow $f^t$ contains a single sink $\omega$ and a single source $\alpha$ in its boundary, and the whole cell is the union of trajectories going from $\alpha$ to $\omega$. 
Equipping of $\mathcal{M}$-vertex. A triangle region

Let us choose a $t$-curve in each cell $J$ which is some usual trajectory in $J$. Let us call an $u$-curve an unstable saddle separatrix with a sink in its closure, an $s$-curve a stable saddle separatrix with a source in its closure. We will call a triangle region $\Delta$ the connecting component of $\bar{M}$.

**Proposition (Oshemkov-Sharko [6])**

Every triangle region $\Delta$ is homeomorphic to an open disk and its boundary consists of an unique $t$-curve, an unique $u$-curve and an unique $s$-curve.
The three-colour graph for a flow

We say that a three-colour graph $\Gamma_M$ corresponds to $f^t$ if:
1) the vertices of $\Gamma_M$ bijectively correspond to the triangle regions of $\Delta_f^t$;
2) two vertices of $\Gamma_M$ are incident to an edge of colour $s$, $t$ or $u$ if the polygonal regions corresponding to these vertices has a common $s$-, $t$- or $u$–curve; that establishes an one-to-one correspondence between the edges of $\Gamma_M$ and the colour curves;

Definition

We say that the graph $\Gamma_M$ is the three-colour graph of the flow $f^t$ corresponding to $\phi^t|_M$. 
A flow and its three-colour graph
Equipment of some directed edges

Let us denote by $\pi_{f^t}$ the correspondence described above between elements of $f^t$ and $\Gamma_M$. Let $ut$, $st$- and $su$ cycles be the cycles of $\Gamma_M$ consisting only of the edges of corresponding colours.

**Proposition**

The projection $\pi_{f^t}$ gives an one-to-one correspondence between the sets $\Omega_{f^t}^0$, $\Omega_{f^t}^1$, $\Omega_{f^t}^2$ and the sets of $tu$-cycles, $su$-cycles of the length 4, and $st$-cycles respectively.

By our construction $M = M \cup D$ each connected component of $D$ contains one sink $\omega$ (source $\alpha$) corresponding to $R_c$ for $c$ of $\phi^t$, which corresponds to an $(\mathcal{M}, \mathcal{L})$-edge ($(\mathcal{L}, \mathcal{M})$-edge) of $\Upsilon_{\phi^t}$. Thus we induce an orientation from $R_c$ to the cycle.
The equipped graph

**Definition**

Let $\Upsilon_{\phi^t}$ be the directed graph of a flow $\phi^t \in G$. We will say that $\Upsilon_{\phi^t}$ is the **equipped graph** of $\phi^t$ and denote it by $\Upsilon^*_{\phi^t}$ if:

1. every $\mathcal{M}$-vertex is equipped with a four-colour graph $\Gamma_{\mathcal{M}}$ corresponding to the flow $f^t$ constructed before;

2. every edge $((\mathcal{M}, \mathcal{L}))$ is equipped with an oriented $tu$-cycle ($st$-cycle) $\tau_{\mathcal{M},\mathcal{L}}$ ($\tau_{\mathcal{L},\mathcal{M}}$) of $\Gamma_{\mathcal{M}}$ corresponding to the limit cycle $c$ of $\mathcal{L}$ and oriented consistently with $R_c$.

3. every $\mathcal{L}$-vertex is equipped with the cycle modulus $T_c$. 
An example of the equipped graph construction
The classification result

**Definition**

Equipped graphs $\Upsilon^*_\phi^t$ and $\Upsilon^*_\phi''^t$ are said to be isomorphic if there is an one-to one correspondence $\xi$ between all edges and vertices of $\Upsilon^*_\phi^t$ and all edges and vertices of $\Upsilon^*_\phi''^t$ preserving their equipments in the following way:

1. the cycle moduli of vertices $\mathcal{L}$ and $\xi(\mathcal{L})$ are equal;
2. for vertices $\mathcal{M}$ and $\xi(\mathcal{M})$, there is an isomorphism $\psi_\mathcal{M}$ of the three-colour graphs $\Gamma_\mathcal{M}, \Gamma_{\xi(\mathcal{M})}$ such that $\psi_\mathcal{M}(\tau_{\mathcal{M},\mathcal{L}}) = \tau_{\xi(\mathcal{M}),\xi(\mathcal{L})}$ and the orientations of $\psi_\mathcal{M}(\tau_{\mathcal{M},\mathcal{L}})$ and $\tau_{\xi(\mathcal{M}),\xi(\mathcal{L})}$ coincide (similarly for $\tau_{\mathcal{L},\mathcal{M}}$).

**Theorem**

*Flows $\phi^t, \phi''^t \in G$ are topologically conjugate if and only if the equipped graphs $\Upsilon^*_\phi^t$ and $\Upsilon^*_\phi''^t$ are isomorphic.*
References

THANKS FOR YOUR ATTENTION