A proof of a conjecture of Elbert and Laforgia on the zeros of cylinder functions

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Cylinder functions

We define for $z \in \mathbb{C} \setminus (-\infty, 0]$, $\nu \in \mathbb{C}$, and $0 \leq \alpha < 1$, the ***cylinder function*** $\mathscr{C}_\nu(z)$ of order ν by

$$\mathscr{C}_\nu(z) \stackrel{\text{def}}{=} J_\nu(z)\cos(\pi\alpha) + Y_\nu(z)\sin(\pi\alpha).$$

Here, $J_\nu(z)$ and $Y_\nu(z)$ denote the ***Bessel functions of the first and second kind***, respectively. These functions are defined by

$$J_\nu(z) \stackrel{\text{def}}{=} \left(\tfrac{1}{2}z\right)^\nu \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}z\right)^{2n}}{\Gamma(\nu+n+1)n!}$$

and

$$Y_\nu(z) \stackrel{\text{def}}{=} \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)},$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\nu \in \mathbb{C}$. When ν is an integer, the limiting value has to be taken in the definition of $Y_\nu(z)$.

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Large zeros of cylinder functions

Assume that ν is real and $-\frac{1}{2} < \nu < \frac{1}{2}$. It is known that the cylinder function $\mathscr{C}_\nu(z)$ has an infinite number of positive zeros, which we denote by $j_{\nu,\kappa}$ with $\kappa = k + \alpha > \frac{1}{2}(|\nu| - \nu)$, $k \in \mathbb{N}$.

James McMahon[1] showed in 1894 that, as $k \to +\infty$, the sequence $j_{\nu,\kappa}$ has an asymptotic expansion

$$\begin{aligned} j_{\nu,\kappa} &\sim \beta_{\nu,\kappa} + \sum_{n=1}^{\infty} \frac{c_n(\nu)}{\beta_{\nu,\kappa}^{2n-1}} \\ &= \beta_{\nu,\kappa} - \frac{4\nu^2 - 1}{8\beta_{\nu,\kappa}} - \frac{(4\nu^2-1)(28\nu^2-31)}{384\beta_{\nu,\kappa}^3} + \cdots \end{aligned}$$

where

$$\beta_{\nu,\kappa} \stackrel{\text{def}}{=} \left(\kappa + \tfrac{1}{2}\nu - \tfrac{1}{4}\right)\pi,$$

and the coefficients $c_n(\nu)$ are polynomials in ν^2 of degree n.

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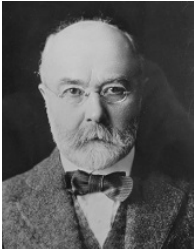
A conjecture of Á. Elbert and A. Laforgia[2]



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Conjecture (Árpád Elbert and Andrea Laforgia, 2001)

For $-\frac{1}{2} < \nu < \frac{1}{2}$, an even (odd) number of terms of McMahon's expansion always gives upper (lower) bounds for $j_{\nu,\kappa}$.

The assumption $\beta_{\nu,\kappa} > 0$ is needed for the statement to be true.

[2] Á. Elbert, A. Laforgia, A conjecture on the zeros of Bessel functions, *J. Comput. Appl. Math.* **133** (2001), no. 1–2, p. 683.

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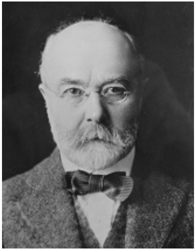

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Known special cases

In the special case $\alpha = 0$, i.e., $\kappa = k = 1, 2, \ldots$, KLAUS-JÜRGEN FÖRSTER and KNUT PETRAS[3] proved in 1993 that

$$\beta_{\nu,k} < j_{\nu,k},$$

$$j_{\nu,k} < \beta_{\nu,k} - \frac{4\nu^2 - 1}{8\beta_{\nu,k}},$$

$$\beta_{\nu,k} - \frac{4\nu^2 - 1}{8\beta_{\nu,k}} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384\beta_{\nu,k}^3} < j_{\nu,k},$$

for any $-\frac{1}{2} < \nu < \frac{1}{2}$. Their derivation is lengthy and complicated (but elementary).

[3] K.-J. Förster, K. Petras, Inequalities for the zeros of ultraspherical polynomials and Bessel functions, *Z. angew. Math. Mech.* **73** (1993), no. 9, pp. 232–236.

Main steps of the proof of the conjecture

- For $-\frac{1}{2} < \nu < \frac{1}{2}$ and $w > 0$, we re-express the cylinder function in the form

$$\mathscr{C}_\nu(z) = \sqrt{J_\nu^2(z) + Y_\nu^2(z)} \cos\left(\Theta_\nu(z) - \left(\alpha + \tfrac{1}{2}\nu + \tfrac{1}{4}\right)\right),$$

  where the phase function $\Theta_\nu(z)$ is normalised so that $\Theta_\nu(j_{\nu,\kappa}) = \beta_{\nu,\kappa}$. Then this function is continued analytically to the right half-plane $\mathfrak{Re} z > 0$.

- We construct a function $X_\nu(w)$ that is analytic in a domain containing the half-plane $\mathfrak{Re} w \geq 0$ and satisfies $X_\nu(\beta_{\nu,\kappa}) = j_{\nu,\kappa}$ for $\beta_{\nu,\kappa} > 0$.

- Employing a Cauchy–Heine trick, we derive an explicit formula for the remainder of McMahon's expansion (the asymptotic expansion of $X_\nu(w)$).

- Appealing to certain properties of $X_\nu(w)$ finishes the proof.

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A function that returns the zeros[4]

Theorem (G. N., 2020)

There exists a function $X_\nu(w)$, with order $-\frac{1}{2} < \nu < \frac{1}{2}$ and argument w, which is analytic in a domain containing the closed half-plane $\mathfrak{Re} w \geq 0$ and has the following properties.

(i) For $\beta_{\nu,\kappa} > 0$, $X_\nu(\beta_{\nu,\kappa}) = j_{\nu,\kappa}$.

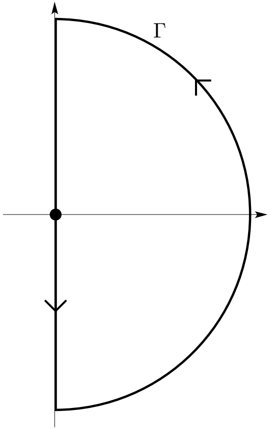
(ii) For any $s > 0$, $\mathfrak{Re} X_\nu(\mathrm{i}s) > 0$.

(iii) $X_\nu(w) = w + \mathcal{O}(w^{-1})$ as $w \to \infty$ in $\mathfrak{Re} w \geq 0$.

(iv) $\mathfrak{Re} X_\nu(\mathrm{i}s) = o(s^{-r})$ as $s \to +\infty$, with any fixed $r > 0$.

[4] G. Nemes, Proofs of two conjectures on the real zeros of the cylinder and Airy functions, *SIAM J. Math. Anal.*, accepted

Contour of integration



The contour Γ used to prove the explicit remainder term.

Truncated expansion with explicit remainder

Let Γ be a D-shaped contour depicted in the previous slide. By the Cauchy integral formula

$$X_\nu(w) - w = \frac{1}{2\pi \mathrm{i}}\oint_\Gamma \frac{X_\nu(t)-t}{t-w}\mathrm{d}t - \underbrace{\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \frac{X_\nu(t)-t}{t+w}\mathrm{d}t}_{0}$$

for any $w > 0$ inside Γ. By blowing up the contour, we obtain

$$\begin{aligned} X_\nu(w) - w &= \frac{1}{2\pi \mathrm{i}}\int_{+\mathrm{i}\infty}^{0} \frac{X_\nu(t)-t}{t-w}\mathrm{d}t + \frac{1}{2\pi \mathrm{i}}\int_{0}^{-\mathrm{i}\infty} \frac{X_\nu(t)-t}{t-w}\mathrm{d}t \\ &\quad - \frac{1}{2\pi \mathrm{i}}\int_{+\mathrm{i}\infty}^{0} \frac{X_\nu(t)-t}{t+w}\mathrm{d}t - \frac{1}{2\pi \mathrm{i}}\int_{0}^{-\mathrm{i}\infty} \frac{X_\nu(t)-t}{t+w}\mathrm{d}t \\ &= \frac{1}{w}\frac{2}{\pi}\int_0^{+\infty} \frac{\mathfrak{Re} X_\nu(\mathrm{i}s)}{1+(s/w)^2}\mathrm{d}s \end{aligned}$$

for any $w > 0$, where use has been made of the fact that

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for any $r > 0$ as $s \to +\infty$. Therefore, for any positive integer N, $w > 0$ and $s > 0$, we can use the expansion

$$\frac{1}{1+(s/w)^2} = \sum_{n=1}^{N-1} \frac{1}{w^{2n-2}}(-1)^{n-1}s^{2n-2} + \frac{1}{w^{2N-2}}(-1)^{N-1}\frac{s^{2N-2}}{1+(s/w)^2},$$

to deduce

$$X_\nu(w) = w + \sum_{n=1}^{N-1} \frac{c_n(\nu)}{w^{2n-1}} + \frac{1}{w^{2N-1}}(-1)^{N-1}\frac{2}{\pi}\int_0^{+\infty} \frac{s^{2N-2}\mathfrak{Re} X_\nu(\mathrm{i}s)}{1+(s/w)^2}\mathrm{d}s$$

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$$0 < \frac{1}{1 + (s/w)^2} < 1$$

for any $w > 0$ and $s > 0$, by the first mean value theorem for improper integrals, there is a $0 < \delta_{\nu,w,N} < 1$ such that

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The final result

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For any $-\frac{1}{2} < \nu < \frac{1}{2}$, $w > 0$, and any positive integer N, the function $X_\nu(w)$ admits the expansion

$$X_\nu(w) = w + \sum_{n=1}^{N-1} \frac{c_n(\nu)}{w^{2n-1}} + \delta_{\nu,w,N} \frac{c_N(\nu)}{w^{2N-1}},$$

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