Compactness properties of operator of translation along trajectories in evolution equations

Władysław Klinikowski
(Joint work with A. Ćwiszewski)

Nicolaus Copernicus University in Toruń

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Let us consider following autonomous wave equation:

\[ u_{tt} + \alpha u_t = \Delta u - V(x)u + f(t, x, u), \quad t > 0, \quad x \in \mathbb{R}^N, \]  \hspace{1cm} (1)

where \( \alpha \geq 0 \) is a damping coefficient and \( V \) is the so called Kato-Rellich potential, i.e. \( V = V_\infty - V_0 \), \( V_\infty \in L^\infty(\mathbb{R}^N) \), \( V_0 \in L^p(\mathbb{R}^N) \), \( 2 < p < \infty \). In general, the nonlinear forcing term \( f \) is continuous and locally Lipschitz (currently we work yet by global Lipschitz condition) with respect to the third variable and \( T \)-periodic in time, i.e. \( f(t + T, x, u) = f(t, x, u) \). The two cases must be considered separately: the resonant and non-resonant ones. In the resonant case, that is when the kernel space of the linearization of the \( -\Delta + V(x) + f(t, x, \cdot) \) is nontrivial, i.e.

\[ \mathcal{N} := \text{Ker} (-\Delta + V) \neq \{0\}. \]  \hspace{1cm} (2)
We introduce so-called Landesman-Lazer type conditions, which mean that

\[
\int_0^T \left( \int_{\{\phi>0\}} \tilde{f}^+ (t, x) \phi(x) \, dx + \int_{\{\phi<0\}} \tilde{f}^- (t, x) \phi(x) \, dx \right) \, dt > 0,
\]

for any \( \phi \in \mathcal{N} \setminus \{0\} \), where \( \tilde{f}^+ (t, x) := \liminf_{s \to +\infty} f(t, x, s) \) and \( \tilde{f}^- (t, x) := \limsup_{s \to -\infty} f(t, x, s) \) or

\[
\int_0^T \left( \int_{\{\phi>0\}} \hat{f}^+ (t, x) \phi(x) \, dx + \int_{\{\phi<0\}} \hat{f}^- (t, x) \phi(x) \, dx \right) \, dt < 0,
\]

for any \( \phi \in \mathcal{N} \setminus \{0\} \), where \( \hat{f}^+ (t, x) := \limsup_{s \to +\infty} f(t, x, s) \) and \( \hat{f}^- (t, x) := \liminf_{s \to -\infty} f(t, x, s) \).

It is noteworthy that the conditions of that type can be verified without the explicit knowledge of \( \mathcal{N} \) (see Remark 1.3 in [5]).
We are now in point to state our first theorem

**Theorem 1**

*Suppose that we have problem (1) with assumptions about* $V(x)$ *and* $f(t, x, u)$ *as mentioned above and additionally we request that* $f$ *is bounded by square integrable function, i. e. there exists* $m \in L^2(\mathbb{R}^N)$ *such that for all* $t \in [0, \infty)$, $u \in \mathbb{R}$ *and almost every* $x \in \mathbb{R}^N$

$$|f(t, x, u)| \leq m(x).$$

*Moreover, we require that resonance condition* (2) *holds as well as one Landesman-Lazer conditions* (3), (4). *Next we assume that*

$$a_\infty := \lim_{R \to +\infty} \text{essinf}_{|x| \geq R} V_\infty(x)$$

*is a positive number and* $0 \in \sigma_p(-\Delta + V(x))$. *Then there exists* $T$-*periodic solution of* (1).
The second case concerning situation when there are no resonance, which means that $-\Delta + V(x) + f(t, x, \cdot)$ at zero and infinity has trivial kernels. We shall use the following linearizations of $f$

$$\lim_{u \to 0} \frac{f(t, x, u)}{u} = \alpha(t, x), \quad \lim_{|u| \to \infty} \frac{f(t, x, u)}{u} = \omega(t, x),$$

(5)

for all $x \in \mathbb{R}^N$ and $t \geq 0$, where $\alpha(t, \cdot)$, $\omega(t, \cdot)$ are Kato-Rellich potentials. We work under the assumption that

$$\text{Ker} (-\Delta + V - \hat{\omega}) = \{0\} \quad \text{and} \quad \text{Ker} (-\Delta + V - \hat{\alpha}) = \{0\},$$

(6)

where

$$\hat{\alpha}(x) := \frac{1}{T} \int_0^T \alpha(t, x) \, dt, \quad \hat{\omega}(x) := \frac{1}{T} \int_0^T \omega(t, x) \, dt.$$ 

Now we formulate theorem concerning nonresonance case.

**Theorem 2**

*Suppose that we have problem (1) with assumptions about $V(x)$ and $f(t, x, u)$ as mentioned above. Moreover, we require that nonresonance condition (6) holds and linearizations of right-hand-side of (1) are topologically different, i.e. numbers of the negative eigenvalues (counted with multiplicities) of $-\Delta + V - \hat{\alpha}$ and $-\Delta + V - \hat{\omega}$ are different modulo 2. Then there exists $T$-periodic solution of (1).*
The hyperbolic equation (1) can be rewritten as the system on space $\mathbf{X} = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

\[
\begin{align*}
    u_t &= v - \delta u \\
    v_t &= -(-\Delta + V(x))u - (\alpha - \delta)(v - \delta u) + f(t, x, u)
\end{align*}
\] (7)

with $\delta \geq 0$ (see section 2.1 in [8]). We equip space $\mathbf{X}$ with usual scalar product $(., .)_{\mathbf{X}} := (., .)_{H^1} + (., .)_{L^2}$. 
Subsequently we transform formula above into first order differential equation

\[(u_t, v_t) = -(A_0 + V)(u, v) + F(t, u, v), \quad t > 0, \]

where \(A_0 : D(A_0) \subset X \to X\), is defined by \(D(A_0) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\) and

\[A_0(u, v) := \left(\delta u - v, -\Delta u + (\alpha - \delta)v + (\delta^2 - \alpha\delta)u\right),\]

and \(V : X \to X\) by

\[V(u, v) = (0, V(x)u).\]

The mapping \(F : [0, +\infty) \times X \to X\) is given by

\[F(t, u, v) := (0, F(t, u))\]

with the Nemytskii operator \(F : [0, +\infty) \times H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N)\) given by

\[[F(t, u)](x) := f(t, x, u(x)).\]
It is may be shown, by use of the Lumer-Phillips theorem, that $-(A_0 + V)$ is the generator of $C_0$-semigroup of contractions $\{S_{-(A_0+V)}(t)\}_{t \geq 0}$, which enables us to consider mild solutions of (8), i.e. we say that function $(u(.), v(.)) : [0, \tau] \to X$ is a mild solution with initial condition $(\bar{u}, \bar{v})$ if it satisfies following integral formula

$$(u(t), v(t)) = S_{-(A_0+V)}(t)(\bar{u}, \bar{v}) + \int_0^t S_{-(A_0+V)}(t-s)F(s, u(s), v(s)) \, ds, \quad (9)$$

for any $t \in [0, \tau]$. Consequently, we say that $u : [0, \tau] \to H^1(\mathbb{R}^N)$ is a mild solution of (1) if there exists $v : [0, \tau] \to L^2(\mathbb{R}^N)$ such that $(u, v)$ is a mild solution of (8). In particular, under reasonable assumptions on $f$, we have the Poincaré operator of translation along trajectories $\Phi_T : X \to X$ associated with (8), defined by

$$\Phi_T(\bar{u}, \bar{v}) := (u(T), v(T)),$$

where $(u, v) : [0, T] \to X$ is the mild solution of (8) with the initial condition $(u(0), v(0)) = (\bar{u}, \bar{v})$ (the existence and uniqueness come from standard $C_0$-semigroup theory, see e.g. [11] or [3]).

Observe that if $(\bar{u}, \bar{v}) \in X$ is a fixed point of $\Phi_T$, i.e. $\Phi_T(\bar{u}, \bar{v}) = (\bar{u}, \bar{v})$, then the mild solution of (8) is $T$-periodic. Therefore we search for fixed points of $\Phi_T$. 
Let $(E, \|\cdot\|)$ be a Banach space. We say that $G : E \to E$ is *ultimately compact* if for any bounded $D \subset E$ such that

$$D \subset \overline{\text{conv}}(G(D)),$$

then $D$ is relatively compact in $E$. Observe that any compact map is ultimately compact. We recall that $\text{conv}(D)$ means convex hull of set $D$ and $\overline{\text{conv}}(D)$ means convex closed hull of $D$ (we consider closure in space $E$). There are possible slightly general definitions, for instance see section 2 in [6]. We are in a position to introduce index theory for ultimately compact sets due to Sadovskii (see Section 3.5.6 in [1]). It is an improvement of index for so-called condensing operators due to Nussbaum and Sadovskii (see Chapter 4 in [10]). One can show that index for ultimately compact operator possesses standard properties (existence, additivity, homotopy invariance and normalization). Moreover, it is also true, that if $G$ is a compact map, then index for ultimately compact operators coincides with Leray-Schauder index. Further we will denote index of ultimate compact map $G$ with respect to open set $U \subset E$ by $\text{Ind}_{uc}(G, U)$. 
According to what was said before we need to prove that Poincaré operator of translation along trajectories is ultimately compact. One of techniques is based on the so-called tail estimates of solutions, for instance estimation of sequences

$$\int_{\{|x| \geq n\}} |u(x, t)|^2 \, dx,$$

where $u : [0, \infty) \to H^1(\mathbb{R}^N)$ is a solution of parabolic equation

$$u_t = \Delta u - V(x)u + f(t, x, u), \quad t > 0, \quad x \in \mathbb{R}^N. \quad (10)$$

The tail estimate method was originated for parabolic equations by Wang in [13]. A tail estimates enabling applications of Conley index to parabolic equation and the associated semiflow on $H^1(\mathbb{R}^N)$ was shown in [12] and adapted in [5]. We shall follow ideas from [6] and [7] for parabolic problems (10) and from [8] for hyperbolic problems.
Assume now that, for all \( t \geq 0, x \in \mathbb{R}^N \) and \( u_1, u_2 \in \mathbb{R} \),

\[
|f(t, x, u_1) - f(t, x, u_2)| \leq l(x)|u_1 - u_2|
\]

where \( l \) is a Rellich-Kato type function. Let \( u_1, u_2 \) be two solution of (8). Then \( u := u_1 - u_2 \) and \( v := \delta u + \dot{u} \) satisfy

\[
\dot{u} = \Delta u - ((\alpha - \delta)\delta - V)u - (\alpha - \delta)u + F(t, u_1) - F(t, u_2)
\]

Furthermore, \( \phi : [0, \infty) \to \mathbb{R} \) be smooth function such that \( \phi([0, \infty)) \subset [0, 1] \), for any \( s \in [0, 1] \) \( \phi(s) = 0 \) and for any \( s \in [2, \infty) \) \( \phi(s) = 1 \). Then for any \( k \in \mathbb{N}_{\geq 1} \) we put \( \phi_k : \mathbb{R}^N \to \mathbb{R} \), \( \phi_k(x) := \phi(|x|^2/k^2) \), where \(|.|\) stands for norm in \( \mathbb{R}^N \). By the regularity theory (see [2])

\[
\frac{1}{2} \frac{d}{dt} (v, v\phi_k)_0 = (\dot{v}, v\phi_k)_0 = (\Delta u, v\phi_k)_0 + (((\alpha - \delta)\delta - V)u, v\phi_k)_0 +
\]

\[
-(\alpha - \delta)(v, v\phi_k)_0 + (F(t, u_1) - F(t, u_2), v\phi_k)_0
\]

\[
= I_1 + I_2 + I_3
\]

with

\[
I_1 = (\Delta u, v\phi_k)_0, \quad I_2 = (((\alpha - \delta)\delta - V)u, v\phi_k)_0, \quad I_3 = -(\alpha - \delta)(v, v\phi_k)_0 + (F(t, u_1) - F(t, u_2), v\phi_k)_0.
\]
Clearly

\[ I_1 \leq -\frac{1}{2} \frac{d}{dt} (\nabla u, \nabla u\phi_k)_0 - \delta (\nabla u, \nabla u\phi_k)_0 + c_k^{(1)} \]

where \( c_k^{(1)} \to 0^+ \) as \( k \to +\infty \).

Further we have

\[
I_2 = \left( ( (\alpha - \delta)\delta - a_\infty \right) u, \dot{u}\phi_k + \delta u\phi_k)_0 + ((a_\infty - V)u, v\phi_k)_0
\]

\[
= -\frac{1}{2} \frac{d}{dt} ((a_\infty - (\alpha - \delta)\delta)u, u\phi_k)_0 - \delta ((a_\infty - (\alpha - \delta)\delta)u, u\phi_k)_0 +
\]

\[
+ ((a_\infty - V)u, v\phi_k)_0 + (V_0 u, v\phi_k)_0 \leq
\]

\[
\leq -\frac{1}{2} \frac{d}{dt} ((a_\infty - (\alpha - \delta)\delta)u, u\phi_k)_0 - \delta ((a_\infty - (\alpha - \delta)\delta)u, u\phi_k)_0 + c_k^{(2)},
\]

where

\[ c_k^{(2)} = (V_0 u, v\phi_k)_0. \]

Clearly, by standard tail estimate techniques

\[ c_k^{(2)} \to 0^+ \quad \text{as} \quad k \to +\infty. \]
Finally observe that, that for large $k$,

\[
I_3 = -(\alpha - \delta)(v, v\phi_k)_0 + (F(t, u_1) - F(t, u_2), v\phi_k)_0 \\
\leq -(\alpha - \delta)(v, v\phi_k)_0 + \text{essup}_{|x| \geq k} l_{\infty}(x)(|u|, |v|\phi_k)_0 + (l_0|u|, |v|\phi_k)_0 \\
\leq -(\alpha - \delta)(v, v\phi_k)_0 + (\bar{l}_{\infty}/2) \varepsilon(v, v\phi_0) + (\varepsilon\bar{l}_{\infty}/2) (u, u\phi_k)_0 + c^{(3)}_k
\]

with

\[
\bar{l}_{\infty} = \lim_{R \to +\infty} \text{essup}_{|x| \geq k} l_{\infty}(x)
\]

$c^{(3)}_k \to 0^+$ as $k \to +\infty$. Hence if we put

\[
D(t) := (\nabla u, \nabla u\phi_k)_0 + ((a_{\infty} - (\alpha - \delta)\delta)u, u\phi_k)_0 + (v, v\phi_k)_0
\]

then, by the above estimates we get

\[
\dot{D}(t) \leq -2\delta D(t) - (\alpha - 2\delta)(v, v\phi_k)_0 + \bar{l}_{\infty}/2\varepsilon(v, v\phi_0) + (\varepsilon\bar{l}_{\infty}/2) (u, u\phi_k)_0 + c_k
\]

with $c_k \to 0^+$ as $k \to +\infty$. If we find $\varepsilon > 0$ and $\delta > 0$ such that

\[
-\delta - (\alpha - 2\delta) + \frac{L}{2\varepsilon} < 0 \quad \text{and} \quad \frac{\varepsilon L}{2(a_{\infty} - (\alpha - \delta)\delta)} - \delta < 0 \quad (11)
\]

with $L = \bar{l}_{\infty}$, then we get $\rho > 0$ such that

\[
\dot{D}(t) \leq -\rho D(t) + c_k,
\]
Observe that, for any $\delta \in (0, \alpha)$,

$$(\alpha - \delta)\delta \leq \left(\frac{\alpha}{2}\right)^2$$

and that if we assume that

$$a_\infty > \left(\frac{\alpha}{2}\right)^2$$

and find $\delta > 0$ and $\varepsilon > 0$ such that

$$\delta < \alpha - \frac{L}{2\varepsilon} \quad \text{and} \quad \frac{\varepsilon L}{2(a_\infty - \left(\frac{\alpha}{2}\right)^2)} < \delta$$

then these numbers satisfy (11). It can be shown that such numbers exists if

$$L/2 < \alpha^2(a_\infty - \left(\frac{\alpha}{2}\right)^2).$$

**Theorem 3**

*If $a_\infty$, $\bar{l}_\infty$ and $\alpha$ satisfy

$$\bar{l}_\infty < 2\alpha^2(a_\infty - \left(\frac{\alpha}{2}\right)^2),$$

then the Poincaré operator $\Phi_T : X \to X$ is ultimately compact.*
Once having the ultimate property, we shall be able to perform fixed point index computations for $\Phi_T$ using some geometric properties of the equation. The resonant and non-resonant cases are considered separately. We shall follow the ideas from [6] and [7].

In the non-resonant case we shall consider the following family of equations

$$ (u_t, v_t) = -(A_0 + V)(u, v) + F(t/\varepsilon, u, v), \ t > 0, \quad (12) $$

where $\varepsilon \in (0, 1]$. First we shall show that the non-resonance conditions (6) imply the existence of $r > 0$ and $R > 2r$ such that, for all $\varepsilon \in (0, 1]$, the equation (12) has no nontrivial $\varepsilon T$-periodic solution with

$$ (u(0), v(0)) \in B_X(0, 2r) \cup (X \setminus B_X(0, R)) . $$

This will enable us to consider the translation along trajectories operator $\Phi_T^{(\varepsilon)}$ for (12) and ask for the indices $\text{Ind}_{uc}(\Phi_T^{(\varepsilon)}, B_X(0, r))$ and $\text{Ind}_{uc}(\Phi_T^{(\varepsilon)}, B_X(0, R))$ where $\text{Ind}_{uc}$ denotes fixed point index for ultimately compact mappings. By the homotopy property (note that $\Phi_T = \Phi_T^{(1)}$)

$$ \text{Ind}_{uc}(\Phi_T, B_X(0, r)) = \lim_{\varepsilon \to 0^+} \text{Ind}_{uc}(\Phi_T^{(\varepsilon)}, B_X(0, r)) $$

and

$$ \text{Ind}_{uc}(\Phi_T, B_X(0, R)) = \lim_{\varepsilon \to 0^+} \text{Ind}_{uc}(\Phi_T^{(\varepsilon)}, B_X(0, R)). $$
Then the averaging principle (see e.g. [9] or [4]) will be used to show that, for \( U = B_X(0, r) \) and \( U = B_X(0, R) \), there exists \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0] \), one has

\[
\text{Ind}_{uc}(\Phi^{(\varepsilon)}_T, U) = \text{Ind}_{uc}(\hat{\Phi}_{\varepsilon T}, U)
\]

where \( \hat{\Phi}_t, t > 0, \) are the Poincaré operators associated with

\[
\begin{align*}
\dot{u} + \alpha u &= \Delta u - V(x)u + \hat{f}(x, u), \quad t > 0, \ x \in \mathbb{R}^N, \\
\end{align*}
\]  

that is with

\[
(u_t, v_t) = -(A_0 + V)(u, v) + \hat{F}(u, v), \quad t > 0, 
\]  

where \( \hat{f}(x, u) := \frac{1}{T} \int_0^T f(t, x, u) \, dt \) and \( [\hat{F}(u, v)](x) := (0, \hat{f}(x, u(x))) \). Hence, to find the indices

\[
\text{Ind}_{uc}(\Phi_T, B_X(0, r)) \text{ and } \text{Ind}_{uc}(\Phi_T, B_X(0, R))
\]

it sufficient to compute the fixed point index of \( \hat{\Phi}_t \) for small \( t > 0 \).
To get the fixed point index of $\hat{\Phi}_t$ we shall use the linearization method and the spectral properties of operators $-\Delta + V - \hat{\alpha}$ and $-\Delta + V - \hat{\omega}$. Namely, we expect that there exists $t_0 > 0$ such that, for all $t \in (0, t_0]$,

$$\text{Ind}_{uc}(\Phi_T, B_X(0, r)) = (-1)^{m(0)} \quad \text{and} \quad \text{Ind}_{uc}(\Phi_T, B_X(0, R)) = (-1)^{m(\infty)}$$

where $m(0)$ is the number of negative eigenvalues of $-\Delta + V - \hat{\alpha}$ (counted with their multiplicities) and $m(\infty)$ is the number of negative eigenvalues of $-\Delta + V - \hat{\omega}$. Here we use the Weyl spectral theorem to see that the essential spectra of $-\Delta + W_\infty$ and $-\Delta + W_\infty + W_0$ coincide if $W_\infty + W_0$ is a Kato-Rellich potential. Finally, by means of the additivity property of fixed point index, we shall arrive at

$$\text{Ind}_{uc}(\Phi_T, B_X(0, R) \setminus B_X(0, r)) = (-1)^{m(\infty)} - (-1)^{m(0)},$$

i.e. the formula showing the assertion, i.e. if $m(0) \not\equiv m(\infty) \mod 2$, then $\text{Ind}_{uc}(\Phi_T, B_X(0, R) \setminus B_X(0, r)) \neq 0$, which implies the existence of $T$-periodic solution starting from $B_X(0, R) \setminus B_X(0, r)$. 
If equation (1) is at resonance, that is $\mathcal{N} = \ker(-\Delta + V) \neq \{0\}$, then we shall consider the following parameterized equation

$$u_{tt} + \alpha u_t = \Delta u - V(x)u + \varepsilon f(t, x, u), \ t > 0, \ x \in \mathbb{R}^N$$

(15)

and the associated first order problem

$$(u_t, v_t) = -(A_0 + V)(u, v) + \varepsilon F(t, (u, v)), \ t > 0$$

with associated operator of translation $\Phi^{(\varepsilon)}_T$. Let $P : L^2(\mathbb{R}^N) \to \mathcal{N}$ be the orthogonal projection onto $\mathcal{N}$. Observe that $\mathcal{N} \subset H^1(\mathbb{R}^N)$ and, in view of the Riesz-Schauder theory, $\dim \mathcal{N} < \infty$. Then $L^2(\mathbb{R}^N) = \mathcal{N} \oplus \mathcal{N}^\perp$ and consequently $H^1(\mathbb{R}^N) = \mathcal{N} \oplus (\mathcal{N}^\perp \cap H^1(\mathbb{R}^N))$. Subsequently we put

$$\bar{F}(u) = \frac{1}{T} \int_0^T P F(t, u) \, dt.$$  

We expect that if $U \subset \mathcal{N}$ is an open bounded set such that $\bar{F}^\prime(\bar{u}) \neq 0$ for any $\bar{u} \in \partial U$, then for any $r, R > 0$ exists $\varepsilon_0 \in (0, 1]$ such that for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\text{Ind}_{uc} (\Phi^{\varepsilon}_T, (U \oplus B_r) \times B(0, R)) = (-1)^{m(\infty) + \dim \mathcal{N}} \text{Deg}_B(\bar{F}, U),$$

where $B_r = \{u \in \mathcal{N}^\perp \cap H^1(\mathbb{R}^N) \mid \|u\|_{H^1} < r\}$, $B(0, R)$ is an open ball in $L^2(\mathbb{R}^N)$ centered in 0 with radius $R$, $m(\infty)$ is a number of negative eigenvalues (counted with multiplicities) of $-\Delta + V$ and $\text{Deg}_B$ denotes topological Brouwer degree.
Next step of our reasoning should be proving resonance version of continuation principle: if there exists $R_0 > 0$ such that $\text{Deg}_B(\bar{F}, B_{\mathcal{N}}(0, R_0)) \neq 0$ ($B_{\mathcal{N}}(0, R_0) = \{u \in \mathcal{N} \mid \|u\|_{H^1} < R_0\}$) and for any $\varepsilon \in (0, 1)$ there are not $T$-periodic solutions with $\|z(0)\|_X \geq R_0$ of (15), then equation

$$z_t = -(A_0 + V)z + F(t, z), \ t > 0$$

has a $T$-periodic solution. Hence a crucial point is to show that Brouwer degree associated with $\bar{F}$ is nontrivial. We suppose that Landesman-Lazer conditions allow us to provide such result, and, in consequence, an existence of $T$-periodic solutions of equations in resonance.


