Fractional Integrals with Measure in Grand Lebesgue and Morrey spaces

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8ECM, Harmonic Analysis and PDEs, Portorož, Slovenia, June 23

Hardy–Littlewood-Sobolev Inequality

Theorem (HLS). *Let* $0 < \alpha < n$, $1 < p < p^* < \infty$. *Suppose that* $\frac{1}{p} - \frac{1}{p^*} = \frac{\alpha}{n}$. *Then there is a positive constant* C *such that*

$$\|K_\alpha f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n),$$

where

$$K_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Quasi-metric measure space

Let (X, d, μ) be a space of non-homogeneous type, i.e., (X, d, μ) be a topological space endowed with a locally finite complete measure μ and quasi-metric $d : X \times X \mapsto \mathbb{R}_+$ satisfying the following conditions:
$i)$ $d(x, y) = 0$ if and only if $x = y$;
$ii)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;
$iii)$ there exist a constant $\kappa \geq 1$ such that for all $x, y, z \in X$,

$$d(x, y) \leq \kappa[d(x, z) + d(z, y)];$$

(iv) for every neighbourhood N of a point $x \in X$ there exists $r > 0$ such that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in N.
It is also assumed that all balls $B(x, r)$ in X are measurable, and that $\mu\{x\} = 0$ for all $x \in X$.

Fractional Integral Operator

Let

$$(I_\gamma f)(x) = \int\limits_X \frac{f(y)}{d(x,y)^{1-\gamma}} d\mu(y), \;\; 0 < \gamma < 1, \;\; x \in X,$$

be fractional integral with a measure μ.
Taking, for example, new quasi-metric $d_1(x,y) = d(x,y)^{1/n}$, $n > 0$, then we can rewrite $I_\gamma f$ as follows;

$$(T_\alpha f)(x) = \int\limits_X \frac{f(y)}{d_1(x,y)^{n-\alpha}} d\mu(y), \;\; 0 < \alpha < n, \;\; x \in X,$$

where $\alpha = \gamma n$. Thus we have fractional integral operator defined on (X, d_1, μ).

Potentials with measure. Formulation of the problem

Let (X, d, μ) be a non-homogeneous space. Let $1 < p < q < \infty$ and let $0 < \gamma < 1$. To give a complete characterization of a measure μ such that the inequality

$$\|I_\gamma f\|_{L^q(X,\mu)} \leq C\|f\|_{L^p(X,\mu)}, \quad f \in L^p(X, \mu),$$

holds.

Potentials with measure. HLS- type inequality

The following theorem was proved in 2001 in [V. Kokilashvili and A.M. 2001] (For Euclidean spaces see V.Kokilashvili: 1992).

Theorem A. *Let* $1 < p < q < \infty$ *and let* $0 < \gamma < 1$. *Then the inequality*

$$\|I_\gamma f\|_{L^q(X,\mu)} \le C\|f\|_{L^p(X,\mu)}, \quad f \in L^p(X,\mu),$$

holds if and only if there exists a positive constant c *such that for all* $x \in X$ *and* $r \in (0,\ diam(X))$,

$$\mu B(x,r) \le cr^\beta, \tag{0.1}$$

where β *is defined as follows:*

$$\beta := \frac{pq(1-\gamma)}{pq+p-q}. \tag{0.2}$$

Potentials with measure. HLS- type inequality

Multilinear characterization: V. Kokilashvili, M. Mastylo and A. M., *JGA*, 2020.

Compactness characterization: V. Kokilashvili, M. Mastylo and A. M., *FCAA*, 2019.

Potential with measure

As a Corollary we heve HLS type inequality (see also J. Garcia-Cuerva and A. E. Gatto, 2003):

Corollary A. *Let* $1 < p < \frac{1}{\gamma}$, *where* $0 < \gamma < 1$. *We set* $p^* := \frac{p}{1-\gamma p}$. *Then the Hardy–Littlewood–Sobolev type inequality*

$$\|I_\gamma f\|_{L^{p^*}(X,\mu)} \le C\|f\|_{L^p(X,\mu)}, \quad f \in L^p(X,\mu),$$

holds if and only if there exists a positive constant c *such that for all* $x \in X$ *and* $r \in (0,\ diam(X))$,

$$\mu B(x,r) \le cr. \tag{0.3}$$

Grand Lebesgue spaces

In 1992 T. Iwaniec and C. Sbordone, in their studies related with the integrability properties of the Jacobian in a bounded open set Ω of $\mathbb{R}^n$, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. A generalized version of these spaces denoted by $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone in 1997.

Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during last years by many authors due to various applications.

Grand Lebesgue spaces

Let θ be a positive number and let $\mu(X) < \infty$. Denote by $L^{p),\theta}(X,\mu)$ the grand Lebesgue space defined by the norm

$$\|f\|_{L^{p),\theta}(X,\mu)} = \sup_{0<\eta\leq p-1} \eta^{\frac{\theta}{p-\eta}} \|f\|_{L^{p-\eta}(X,\mu)},$$

where $L^r(X,\mu)$, $1 \leq r < \infty$, is the classical Lebesgue space with respect to a measure μ, and defined by the norm:

$$\|f\|_{L^r(X,\mu)} = \left(\int_X |f(x)|^r d\mu(x) \right)^{1/r}.$$

Some properties of grand Lebesgue spaces

The grand Lebesgue space $L^{p),\theta}(\Omega)$ is non-reflexive, non-separable and, in general, is non-rearrangement invariant (see, e.g., A. Fiorenza, 2000).

The following properties hold:

(a) $C_0^\infty(\Omega)$ is not dense in $L^{p),\theta}(\Omega)$;

(b) $L^p(\Omega) \hookrightarrow L^{p),\theta}(\Omega) \hookrightarrow L^{p-\varepsilon}(\Omega)$;

(c) for example, the function $x^{-1/p}$ belongs to $L^{p),1}((0,1)) \setminus L^p((0,1))$;

(d) elements of the closure of $C_0^\infty(\Omega)$ in $L^{p),\theta}(\Omega)$ are characterized by the following property: $\lim\limits_{\varepsilon \to 0} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega)} = 0$.

Our main statement reads as follows:

Theorem

Let $\mu(X) < \infty$, $1 < p < q < \infty$ *and Let* $0 < \gamma < 1$. *Suppose that* $\theta > 0$. *Then* I_γ *is bounded from* $L^{p),\theta}(X,\mu)$ *to* $L^{q),\frac{q\theta}{p}}(X,\mu)$ *if and only if there is a positive constant c such that*

$$\mu B(x,r) \leq cr^{\beta},$$

holds for all $x \in X$ *and* $r \in (0,\ diam(X))$, *where* β *is defined by (0.2), i.e.*

$$\beta := \frac{pq(1-\gamma)}{pq+p-q}.$$

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Generalization of the Sobolev Inequality

Corollary

Let $\mu(X) < \infty$, $1 < p < \infty$ *and let* $0 < \gamma < \frac{1}{p}$. *We set* $p^* = \frac{p}{1-\gamma p}$. *Suppose that* $\theta > 0$. *Then there is a positive constant* C *such that for all* $f \in L^{p),\theta}(X,\mu)$, *the inequality*

$$\|I_\gamma f\|_{L^{p^*),\frac{p^*\theta}{p}}(X,\mu)} \leq C\|f\|_{L^{p),\theta}(X,\mu)}$$

holds if and only if holds if there exists a positive constant c *such that for all* $x \in X$ *and* $r \in (0,\ diam(X))$,

$$\mu B(x,r) \leq cr. \tag{0.4}$$

The sharpness of the second parameter

Proposition *Let* $1 < p < q < \infty$ *and* $0 < \gamma < 1$. *Suppose that* (X, d, μ) *be a non-homogeneous space. Let there exist a positive constant* b *such that for all* $x \in X$ *and* $r \in (0,\ diam\,(X))$,

$$\mu(B(x,r)) \geq br^{\beta}, \tag{0.5}$$

where β *is defined by (0.2). Then the boundedness of* I_γ *from* $L^{p),\theta_1}(X,\mu)$ *to* $L^{q),\theta_2}(X,\mu)$ *implies that* $\theta_2 \geq \frac{\theta_1 q}{p}$.

Grand Morrey spaces

Let (X, d, μ) be a quasi-metric measure space and let $M^{p,r}_{\mu,\ell}(X)$ denote the Morrey space defined with respect to a measure μ which is the class at all measurable functions $f : X \longrightarrow \mathbb{R}$ for which the norm

$$\|f\|_{M^{p,r}_{\mu,\ell}(X)} := \sup_{\substack{a\in X\\ t>0}} \frac{1}{t^{(1/p-1/r)\ell}} \|f\|_{L^p_\mu(B(a,t))}$$

$$:= \sup_{\substack{a\in X\\ t>0}} \frac{1}{t^{(1/p-1/r)\ell}} \left(\int\limits_{B(a,t)} |f(y)|^p d\mu(y) \right)^{1/p}$$

is finite, where $1 < p \leq r < \infty$, $\ell > 0$. If $p = r$, then $M^{p,p}_{\mu,\ell}(X)$ coincides with the Lebesgue space $L^p(X, \mu)$. If $\ell = 1$, then $M^{p,r}_{\mu,\ell}(X)$ is denoted by $M^{p,r}_{\mu}(X)$.

Grand Morrey spaces

On the base of $M^{p,r}_{\mu,\ell}$ we introduce grand Morrey space denoted by $M^{p),r,\theta}_{\mu,\ell}(X)$ and defined by the norm

$$\|f\|_{M^{p),r,\theta}_{\mu,\ell}(X)} = \sup_{0<\varepsilon<p-1} \varepsilon^{\theta}\|f\|_{M^{p-\varepsilon,r}_{\mu,\ell}(X)}$$

where $\theta > 0$.

Grand Morrey spaces

Grand Morrey spaces defined on finite measure with doubling condition were introduced by A.M. in 2009 (see H. Rafeiro, 2012 for further generalizations).

Let $1 < s < p \le r < \infty$ and let

$$\mu B(x,r) \le cr^{\ell}.$$

Then the following embeddings hold:

$$L^r_\mu(X) \hookrightarrow M^{p,r}_{\mu,\ell}(X) \hookrightarrow M^{p),r,\theta}_{\mu,\ell}(X) \hookrightarrow M^{s),r,\theta}_{\mu,\ell}(X).$$

If $\mu(X) < \infty$, then

$$M^{p),r,\theta}_{\mu,\ell}(X) \hookrightarrow \mathcal{L}^{p),\theta}_\mu(X),$$

where $\mathcal{L}^{p),\theta}_\mu(X)$ is the grand Lebesgue space defined by the following norm:

$$\|f\|_{\mathcal{L}^{p),\theta}_\mu(X)} = \sup_{0<\varepsilon<p-1} \varepsilon^\theta \|f\|_{L^{p-\varepsilon}(X,\mu)}.$$

Boundedness in grand Morrey spaces

Theorem

Let $1 < p < q < \infty$ *and let* $0 < \gamma < 1$*. Suppose that the condition*

$$\mu B(x, r) \leq c r^{\beta}$$

is satisfied, where β *is defined by (0.2). Suppose that* $1 < r, s < \infty$ *and let*

$$\frac{1}{p} - \frac{1}{r} = \frac{1}{q} - \frac{1}{s}. \tag{0.6}$$

Then I_{γ} *is bounded from* $M_{\mu}^{p),r,\theta}(X)$ *to* $M_{\mu,\beta}^{q),s,\theta}(X)$*.*

The investigation was carried out jointly with V. Kokilashvili. The results are published in [KoMe, 2001].

The work was supported by the Shota Rustaveli National Foundation grant of Georgia (Project No. DI-18-118).

THANK YOU

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