# Atkin-Lehner theory for Drinfeld modular forms

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# Atkin-Lehner theory

Let $k, N \in \mathbb{N}$ and $p \in \mathbb{Z}$ a prime.

Let $S_k(\Gamma_0(N))$ be the $\mathbb{C}$-vector space of cusp forms of level $N$ and weight $k$.

Let $\mathbf{T}_p$ be the Hecke operator if $p \nmid N$, and let $\mathbf{U}_p$ be the Atkin-Lehner operator if $p|N$.

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If $M|N$, we observe that $\Gamma_0(N) \subset \Gamma_0(M) \implies S_k(\Gamma_0(M)) \subset S_k(\Gamma_0(N))$. Then, forms in $S_k(\Gamma_0(N))$ can be divided in

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All cusp forms coming from a lower level.

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The orthogonal complement of oldforms wrt the Petersson inner product.

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Consequences:

- $\mathbf{U}_p$ eigenvalues have slope, i.e. $p$-adic valuation, $< k-1$ in case of oldforms and $k/2-1$ in case of newforms;
- Gouvêa-Mazur conjectures, Coleman families and much more.

Let $q = p^r$ for a fixed prime $p \in \mathbb{Z}$.

| | |
|---|---|
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| $K = \mathbb{F}_q(t)$ | $\mathbb{Q}$ |
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| $\Omega := \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(K_\infty)$ | $\mathbb{H}$ |
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For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(K_\infty)$, $k, m \in \mathbb{Z}$ and $f : \Omega \to \mathbb{C}_\infty$, we define

$$(f|_{k,m}\gamma)(z) := f(\gamma z)(\det\gamma)^m(cz+d)^{-k}.$$

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**Definition**

A rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ is called a *Drinfeld modular form of weight $k$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ for $\Gamma$* if

- $f$ is holomorphic on $\Omega$ and at all cusps;
- $(f|_{k,m}\gamma)(z) = f(z) \quad \forall \gamma \in \Gamma$.

A Drinfeld modular form $f$ is called a *cusp form* if it vanishes at all cusps.

- We denote by $M_{k,m}(\Gamma_0(\mathfrak{m}))$ and $S_{k,m}(\Gamma_0(\mathfrak{m}))$ the finite dimensional $\mathbb{C}_\infty$-vector spaces of Drinfeld modular forms and Drinfeld cusp forms of weight $k$, type $m$ and level $\mathfrak{m}$.
- From now on $\mathfrak{m} = (\pi)$, $\mathfrak{p} = (P)$ with $\pi, P \in A$ monic and $P$ irreducible of degree $d$ with $(\pi, P) = 1$.
- We have Hecke operators $\mathbf{T}_\mathfrak{p}$ ($\mathfrak{p} \nmid \mathfrak{m}$) and $\mathbf{U}_\mathfrak{p}$ ($\mathfrak{p}|\mathfrak{m}$) acting on $M_{k,m}(\Gamma_0(\mathfrak{m}))$.

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- Consider the *degeneracy maps*:

$$\begin{aligned} \mathbf{D}_1, \mathbf{D}_\mathfrak{p} : S_{k,m}(\Gamma_0(\mathfrak{m})) &\to S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p})) \\ f &\mapsto f \\ f &\mapsto f|_{k,m} \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

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**Definition**

The space of $\mathfrak{p}$-*oldforms of level* $\mathfrak{m}\mathfrak{p}$, denoted by $S_{k,m}^{\mathfrak{p}-old}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, is the subspace of $S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$ generated by the set

$$\{(\mathbf{D}_1, \mathbf{D}_\mathfrak{p})(f_1, f_2) = \mathbf{D}_1 f_1 + \mathbf{D}_\mathfrak{p} f_2 : (f_1, f_2) \in S_{k,m}(\Gamma_0(\mathfrak{m}))^2\}.$$

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and the *twisted trace map* is

$$\begin{aligned} \boldsymbol{Tr}_{\mathfrak{m}}^{\prime\,\mathfrak{m}\mathfrak{p}} : S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p})) &\to S_{k,m}(\Gamma_0(\mathfrak{m})) \\ f &\mapsto \sum_{\gamma \in R_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}} \left(f\,|_{k,m} \left(\begin{smallmatrix} 0 & -1 \\ \pi P & 0 \end{smallmatrix}\right)\right)|_{k,m}\gamma \end{aligned}$$

where $\left(\begin{smallmatrix} 0 & -1 \\ \pi P & 0 \end{smallmatrix}\right)$ is a matrix representing the *Fricke involution* of level $\mathfrak{m}\mathfrak{p}$

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**Definition**

The space of $\mathfrak{p}$-newforms of level $\mathfrak{m}\mathfrak{p}$, denoted by $S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, is given by $Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}) \cap Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\prime\,\mathfrak{m}\mathfrak{p}})$.

**Proposition (Bandini, V. - 2020)**

If $\dim_{\mathbb{C}_\infty} S_{k,m}(GL_2(A)) \leqslant 1$, then $S_{k,m}(\Gamma_0(t))$ is direct sum of oldforms and newforms.

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**Theorem (Bandini, V. - 2020)**

We have that $S_{k,m}(\Gamma_0(\mathfrak{p})) = S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{p})) \oplus S_{k,m}^{\mathfrak{p}-old}(\Gamma_0(\mathfrak{p}))$ if and only if the map $\mathcal{D} := Id - P^{k-2m}(Tr'^{\mathfrak{p}}_{(1)})^2$ is bijective on $S_{k,m}(\Gamma_0(\mathfrak{p}))$.

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**Theorem (Bandini, V. - 2020)**

If the map $\mathcal{D} := Id - (\pi P)^{k-2m}(Tr'^{\mathfrak{m}\mathfrak{p}}_{\mathfrak{m}})^2$ is bijective on $S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, then we have the direct sum decomposition
$S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p})) = S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{m}\mathfrak{p})) \oplus S_{k,m}^{\mathfrak{p}-old}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$

Problem: to get the full equivalence $Ker(\mathcal{D})$ should contain a form $f \neq 0$ and also $Fr^{(\mathfrak{m})}(f)$ for a suitable $f \in S_{k,m}(\Gamma_0(\mathfrak{m}))$.

- Let $\mathfrak{n} = (\nu), \mathfrak{d} = (\delta) \subset A$ be ideals such that $\mathfrak{d} || \mathfrak{n}$. Denote by $W_{\mathfrak{d}}^{\mathfrak{n}}$ a matrix of the form

$$\begin{pmatrix} \delta a & b \\ \nu c & \delta d \end{pmatrix} \quad \text{with} \quad a, b, c, d \in A, \ \delta^2 ad - \nu cb = \zeta\delta \text{ and } \zeta \in \mathbb{F}_q^*.$$

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$$\begin{aligned} \mathbf{W}_{\mathfrak{d}}^{\mathfrak{n}} : S_{k,m}(\Gamma_0(\mathfrak{n})) &\to S_{k,m}(\Gamma_0(\mathfrak{n})) \\ f(z) &\mapsto (f\,|_{k,m} W_{\mathfrak{d}}^{\mathfrak{n}})(z) \end{aligned}$$

for any $W_{\mathfrak{d}}^{\mathfrak{n}}$ as above.

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for any $W_{\mathfrak{d}}^{\mathfrak{n}}$ as above.

- If $\mathfrak{d} = \mathfrak{n}$ we have that $\mathbf{W}_{\mathfrak{n}}^{\mathfrak{n}}$ is the *(full) Atkin–Lehner involution* (Fricke involution) and it can be represented by the matrix

$$W_{\mathfrak{n}}^{\mathfrak{n}} = \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}.$$

**Lemma (V. - 2021)**

Let $\mathfrak{n} = (\nu), \mathfrak{d} = (\delta) \subset A$ be ideals such that $\mathfrak{d} || \mathfrak{n}$. If $f \in S_{k,m}(\Gamma_0(\mathfrak{d}))$ then

$$f\,|_{k,m} \begin{pmatrix} \frac{\nu}{\delta} & 0 \\ 0 & 1 \end{pmatrix} := \mathbf{D}_{\frac{\mathfrak{n}}{\mathfrak{d}}}(f) = f\,|_{k,m} W^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{d}}} \in S_{k,m}(\Gamma_0(\mathfrak{n}))\,.$$

**Lemma (V. - 2021)**

Let $\mathfrak{n} = (\nu), \mathfrak{d} = (\delta) \subset A$ be ideals such that $\mathfrak{d} \| \mathfrak{n}$. If $f \in S_{k,m}(\Gamma_0(\mathfrak{d}))$ then

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$$S^{\mathfrak{p}-old}_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p})) = Span\{\mathbf{W}^{\mathfrak{m}\mathfrak{p}}_1(S_{k,m}(\Gamma_0(\mathfrak{m}))), \mathbf{W}^{\mathfrak{m}\mathfrak{p}}_{\mathfrak{p}}(S_{k,m}(\Gamma_0(\mathfrak{m})))\}.$$

**Lemma (V. - 2021)**

Let $\mathfrak{n} = (\nu), \mathfrak{d} = (\delta) \subset A$ be ideals such that $\mathfrak{d} || \mathfrak{n}$. If $f \in S_{k,m}(\Gamma_0(\mathfrak{d}))$ then

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**Theorem (V. - 2021)**

With assumptions on $\mathfrak{m}$ and $\mathfrak{p}$ as above, let $\mathfrak{d} = (\delta)$ be such that $\delta || \pi$. Then

$$\mathbf{W}^{\mathfrak{m}}_{\mathfrak{d}}(\mathbf{T}_{\mathfrak{p}}(f)) = \mathbf{T}_{\mathfrak{p}}(\mathbf{W}^{\mathfrak{m}}_{\mathfrak{d}}(f)) \text{ if } f \in S_{k,m}(\Gamma_0(\mathfrak{m}))$$
$$\mathbf{W}^{\mathfrak{m}\mathfrak{p}}_{\mathfrak{d}}(\mathbf{U}_{\mathfrak{p}}(f)) = \mathbf{U}_{\mathfrak{p}}(\mathbf{W}^{\mathfrak{m}\mathfrak{p}}_{\mathfrak{d}}(f)) \text{ if } f \in S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$$

- Recall that for $f \in S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, the trace is $\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}(f) = \sum_{\gamma \in R_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}} f|_{k,m}\gamma$.

- Recall that for $f \in S_{k,m}(\Gamma_0(\mathfrak{mp}))$, the trace is $\boldsymbol{Tr}^{\mathfrak{mp}}_{\mathfrak{m}}(f) = \sum_{\gamma \in R^{\mathfrak{mp}}_{\mathfrak{m}}} f|_{k,m}\gamma$.

**Definition**

For a $f \in S_{k,m}(\Gamma_0(\mathfrak{mp}))$ and any divisor $\mathfrak{d}$ of $\mathfrak{mp}$ such that $\mathfrak{d} || \mathfrak{mp}$, we define the *$\mathfrak{d}$-twisted trace map* as

$$\begin{aligned} \boldsymbol{Tr}^{\mathfrak{mp}(\mathfrak{d})}_{\mathfrak{m}} := \boldsymbol{Tr}^{\mathfrak{mp}}_{\mathfrak{m}} \circ \mathbf{W}^{\mathfrak{mp}}_{\mathfrak{d}} : S_{k,m}(\Gamma_0(\mathfrak{mp})) &\to S_{k,m}(\Gamma_0(\mathfrak{m})) \\ f &\mapsto \sum_{\gamma \in R^{\mathfrak{mp}}_{\mathfrak{m}}} (f|_{k,m} W^{\mathfrak{mp}}_{\mathfrak{d}})|_{k,m}\gamma . \end{aligned}$$

- Recall that for $f \in S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, the trace is $\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}(f) = \sum_{\gamma \in R_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}} f|_{k,m}\gamma$.

## Definition

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$$\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\mathfrak{d})} := \boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}} \circ \mathbf{W}_{\mathfrak{d}}^{\mathfrak{m}\mathfrak{p}} : S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p})) \to S_{k,m}(\Gamma_0(\mathfrak{m}))$$
$$f \mapsto \sum_{\gamma \in R_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}} (f|_{k,m} W_{\mathfrak{d}}^{\mathfrak{m}\mathfrak{p}})|_{k,m}\gamma .$$

## Proposition (V. - 2021)

With notations as above, we have:

$$\mathbf{W}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}} \circ \boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\mathfrak{d})} = \begin{cases} \delta^{2m-k} \boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\frac{\mathfrak{m}}{\mathfrak{d}})} & \text{if } \mathfrak{p} \nmid \mathfrak{d} \\ (\frac{\delta}{P})^{2m-k} \boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\frac{\mathfrak{m}\mathfrak{p}^2}{\mathfrak{d}})} & \text{if } \mathfrak{p} | \mathfrak{d} \end{cases} .$$

- Recall that for $f \in S_{k,m}(\Gamma_0(\mathfrak{mp}))$, the trace is $\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}}(f) = \sum_{\gamma \in R_{\mathfrak{m}}^{\mathfrak{mp}}} f|_{k,m}\gamma$.

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## Proposition (V. - 2021)

With notations as above, we have:

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The above proposition implies

$$Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}(\mathfrak{mp})}) = Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}(\mathfrak{p})}).$$

### Corollary

The space of $\mathfrak{p}$-newforms of level $\mathfrak{m}\mathfrak{p}$, denoted by $S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, is given by $Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}) \cap Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\mathfrak{p})})$.

## Corollary

The space of $\mathfrak{p}$-newforms of level $\mathfrak{mp}$, denoted by $S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{mp}))$, is given by $Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}}) \cap Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}(\mathfrak{p})})$.

## Theorem (V. - 2021)

The map $\mathcal{D} := Id - P^{k-2m}(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}(\mathfrak{p})})^2$ is bijective on $S_{k,m}(\Gamma_0(\mathfrak{mp}))$ if and only if we have the direct sum decomposition $S_{k,m}(\Gamma_0(\mathfrak{mp})) = S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{mp})) \oplus S_{k,m}^{\mathfrak{p}-old}(\Gamma_0(\mathfrak{mp}))$.

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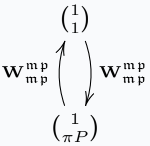
## Action on cusps

Level $\mathfrak{m}\mathfrak{p}$



## Corollary

The space of $\mathfrak{p}$-newforms of level $\mathfrak{mp}$, denoted by $S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{mp}))$, is given by $Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}}) \cap Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}(\mathfrak{p})})$.

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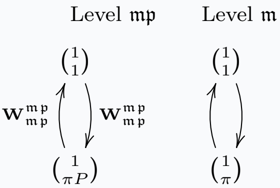

## Corollary

The space of $\mathfrak{p}$-newforms of level $\mathfrak{m}\mathfrak{p}$, denoted by $S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, is given by $Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}) \cap Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\mathfrak{p})})$.

## Theorem (V. - 2021)

The map $\mathcal{D} := Id - P^{k-2m}(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\mathfrak{p})})^2$ is bijective on $S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$ if and only if we have the direct sum decomposition $S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p})) = S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{m}\mathfrak{p})) \oplus S_{k,m}^{\mathfrak{p}-old}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$.

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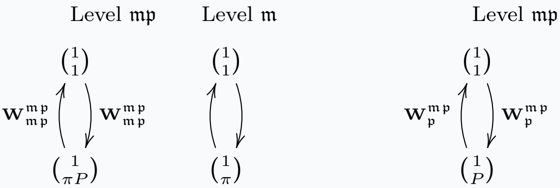


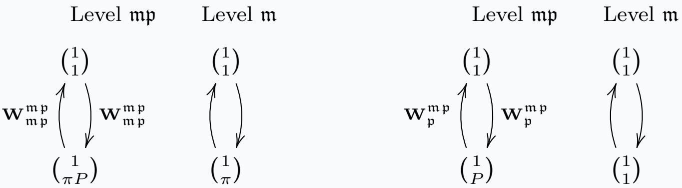
## Corollary

The space of $\mathfrak{p}$-newforms of level $\mathfrak{mp}$, denoted by $S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{mp}))$, is given by $Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}}) \cap Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}(\mathfrak{p})})$.

## Theorem (V. - 2021)

The map $\mathcal{D} := Id - P^{k-2m}(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{mp}(\mathfrak{p})})^2$ is bijective on $S_{k,m}(\Gamma_0(\mathfrak{mp}))$ if and only if we have the direct sum decomposition $S_{k,m}(\Gamma_0(\mathfrak{mp})) = S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{mp})) \oplus S_{k,m}^{\mathfrak{p}-old}(\Gamma_0(\mathfrak{mp}))$.

## Action on cusps



## Corollary

The space of $\mathfrak{p}$-newforms of level $\mathfrak{m}\mathfrak{p}$, denoted by $S_{k,m}^{\mathfrak{p}-new}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$, is given by $Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}}) \cap Ker(\boldsymbol{Tr}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{p}(\mathfrak{p})})$.

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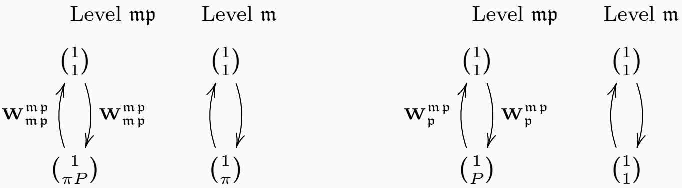


## Proposition (V. - 2020)

The involution $\mathbf{W}_{\mathfrak{p}}^{\mathfrak{m}\mathfrak{p}}$ and the operator $\mathbf{U}_{\mathfrak{p}}$ commute on the space of $\mathfrak{p}$-newforms of level $\mathfrak{m}\mathfrak{p}$.

**Lemma (V. - 2021)**

Let $f \in S_{k,m}(\Gamma_0(\mathfrak{p}))$ be a $\mathfrak{p}$-newform of level $\mathfrak{p}$. Then, $\mathbf{D}_1(f), \mathbf{D}_{\mathfrak{m}}(f) \in S_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$ are $\mathfrak{p}$-newforms of level $\mathfrak{m}\mathfrak{p}$.

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**Proposition (V. 2021)**

Let $f \in M_{k,m}(\Gamma_0(\mathfrak{m}\mathfrak{p}))$ with rational $u$-series coefficients, where $(\mathfrak{m}, \mathfrak{p}) = (1)$ and $\mathfrak{p}$ is prime. Then, $f$ is a $\mathfrak{p}$-adic Drinfeld modular form for $\Gamma_0(\mathfrak{m})$.

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- $f = \sum_{i \geqslant 0} a_i u(z)^i$, $a_i \in A$.
- $v_{\mathfrak{p}}(f) = \inf_i v_{\mathfrak{p}}(a_i)$.

**Lemma (V. - 2021)**

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- $f = \sum_{i \geqslant 0} a_i u(z)^i$, $a_i \in A$.
- $v_{\mathfrak{p}}(f) = \inf_i v_{\mathfrak{p}}(a_i)$.
- We say that $f$ is a *$\mathfrak{p}$-adic Drinfeld modular form* for $\Gamma_0(\mathfrak{m})$ if it exists a sequence $\{f_i\}$ of Drinfeld modular forms for $\Gamma_0(\mathfrak{m})$ verifying $v_{\mathfrak{p}}(f_i - f) \to \infty$ as $i \to \infty$.

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3. M. Valentino *Atkin–Lehner theory for Drinfeld modular forms and applications*, to appear in Ramanujan J., DOI Number 10.1007/s11139-021-00465-0

Thanks for your attention!