Atkin-Lehner theory for Drinfeld modular forms

Maria Valentino

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Let $k, N \in \mathbb{N}$ and $p \in \mathbb{Z}$ a prime.

Let $S_k(\Gamma_0(N))$ be the $\mathbb{C}$-vector space of cusp forms of level $N$ and weight $k$.

Let $T_p$ be the Hecke operator if $p \nmid N$, and let $U_p$ be the Atkin-Lehner operator if $p | N$. 
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Let $T_p$ be the Hecke operator if $p \nmid N$, and let $U_p$ be the Atkin-Lehner operator if $p | N$. If $M | N$, we observe that $\Gamma_0(N) \subset \Gamma_0(M) \implies S_k(\Gamma_0(M)) \subset S_k(\Gamma_0(N))$. Then, forms in $S_k(\Gamma_0(N))$ can be divided in

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All cusp forms coming from a lower level.

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The orthogonal complement of oldforms wrt the Petersson inner product.

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Consequences:

- $U_p$ eigenvalues have slope, i.e. $p$-adic valuation, $< k - 1$ in case of oldforms and $k/2 - 1$ in case of newforms;
- Gouvêa-Mazur conjectures, Coleman families and much more.
Let \( q = p^r \) for a fixed prime \( p \in \mathbb{Z} \).

\[
\begin{array}{c|c|c|c|c|c}
A = \mathbb{F}_q[t] & \mathbb{Z} & \Omega := \mathbb{P}^1(\mathbb{C}_\infty) - \mathbb{P}^1(K_\infty) & \mathbb{H} \\
K = \mathbb{F}_q(t) & \mathbb{Q} & GL_2(A) & SL_2(\mathbb{Z}) \\
K_\infty = \mathbb{F}_q((1/t)) & \mathbb{R} & \Gamma \backslash \mathbb{P}^1(K) & \text{cusps} \\
\mathbb{C}_\infty = \hat{K}_\infty & \mathbb{C} & & \\
\end{array}
\]
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For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K_{\infty})$, $k, m \in \mathbb{Z}$ and $f : \Omega \to \mathbb{C}_{\infty}$, we define

$$(f |_{k,m} \gamma)(z) := f(\gamma z)(\det \gamma)^m (cz + d)^{-k}.$$
Let \( q = p^r \) for a fixed prime \( p \in \mathbb{Z} \).

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A &= \mathbb{F}_q[t] \\
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Fix \( \Gamma = \Gamma_0(m) = \{ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\text{mod } m) \} \), \( m \) ideal of \( A \).
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For \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(K_\infty) \),\( k,m \in \mathbb{Z} \) and \( f : \Omega \to \mathbb{C}_\infty \), we define

\[
(f|_{k,m\gamma})(z) := f(\gamma z)(\det \gamma)^m(cz + d)^{-k}.
\]

Fix \( \Gamma = \Gamma_0(m) = \{ \gamma \equiv \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \pmod{m} \}, m \text{ ideal of } A. \)

**Definition**

A rigid analytic function \( f : \Omega \to \mathbb{C}_\infty \) is called a *Drinfeld modular form of weight \( k \) and type \( m \in \mathbb{Z}/(q - 1)\mathbb{Z} \) for \( \Gamma \) if

- \( f \) is holomorphic on \( \Omega \) and at all cusps;
- \( (f|_{k,m\gamma})(z) = f(z) \ \forall \gamma \in \Gamma. \)

A Drinfeld modular form \( f \) is called a *cusp form* if it vanishes at all cusps.
We denote by $M_{k,m}(\Gamma_0(m))$ and $S_{k,m}(\Gamma_0(m))$ the finite dimensional $\mathbb{C}_\infty$-vector spaces of Drinfeld modular forms and Drinfeld cusp forms of weight $k$, type $m$ and level $m$.

From now on $m = (\pi)$, $p = (P)$ with $\pi, P \in A$ monic and $P$ irreducible of degree $d$ with $(\pi, P) = 1$.

We have Hecke operators $T_p (p + m)$ and $U_p (p|m)$ acting on $M_{k,m}(\Gamma_0(m))$. 
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We have Hecke operators $T_p (p + m)$ and $U_p (p|m)$ acting on $M_{k,m}(\Gamma_0(m))$.

Consider the *degeneracy maps*:

\[
D_1, D_p : S_{k,m}(\Gamma_0(m)) \to S_{k,m}(\Gamma_0(mp)) \\
f \mapsto f \\
f \mapsto f|_{k,m} \left( \begin{array}{cc} P & 0 \\ 0 & 1 \end{array} \right)
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**Definition**

The space of *p-oldforms of level mp*, denoted by $S_{k,m}^{\text{p-old}}(\Gamma_0(mp))$, is the subspace of $S_{k,m}(\Gamma_0(mp))$ generated by the set

$$
\{(D_1, D_p)(f_1, f_2) = D_1 f_1 + D_p f_2 : (f_1, f_2) \in S_{k,m}(\Gamma_0(m))^2\}.
$$
Let $R_{m}^{mp}$ be a set of representatives for $\Gamma_{0}(mp) \backslash \Gamma_{0}(m)$.
Let $R_m^{\text{mp}}$ be a set of representatives for $\Gamma_0(\text{mp}) \backslash \Gamma_0(m)$. The trace map is

$$T_m^{\text{mp}} : S_{k,m}(\Gamma_0(\text{mp})) \to S_{k,m}(\Gamma_0(m))$$

$$f \mapsto \sum_{\gamma \in R_m^{\text{mp}}} f |_{k,m} \gamma.$$
Let $R_{m}^{mp}$ be a set of representatives for $\Gamma_{0}(mp) \backslash \Gamma_{0}(m)$. The **trace map** is

$$Tr_{m}^{mp} : S_{k,m}(\Gamma_{0}(mp)) \to S_{k,m}(\Gamma_{0}(m))$$

$$f \mapsto \sum_{\gamma \in R_{m}^{mp}} f |_{k,m} \gamma.$$ 

and the **twisted trace map** is

$$Tr'_{m}^{mp} : S_{k,m}(\Gamma_{0}(mp)) \to S_{k,m}(\Gamma_{0}(m))$$

$$f \mapsto \sum_{\gamma \in R_{m}^{mp}} (f |_{k,m} \left( \begin{array}{cc} 0 & 1 \\ \pi P & 0 \end{array} \right) \right) |_{k,m} \gamma$$

where $\left( \begin{array}{cc} 0 & 1 \\ \pi P & 0 \end{array} \right)$ is a matrix representing the **Fricke involution** of level $mp$

$$Fr^{(mp)} : S_{k,m}(\Gamma_{0}(mp)) \to S_{k,m}(\Gamma_{0}(mp))$$

$$f \mapsto f |_{k,m} \left( \begin{array}{cc} 0 & -1 \\ \pi P & 0 \end{array} \right)$$
Let $R_{m}^{mp}$ be a set of representatives for $\Gamma_{0}(mp) \backslash \Gamma_{0}(m)$.

The **trace map** is

$$ Tr_{m}^{mp} : S_{k,m}(\Gamma_{0}(mp)) \rightarrow S_{k,m}(\Gamma_{0}(m)) $$

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and the **twisted trace map** is

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**Definition**

The space of $p$-newforms of level $mp$, denoted by $S_{k,m}^{p-new}(\Gamma_{0}(mp))$, is given by

$Ker(Tr_{m}^{mp}) \cap Ker(Tr_{m}^{\prime}mp)$. 
Proposition (Bandini, V. - 2020)

If \( \dim_{\mathbb{C}_\infty} S_{k,m}(GL_2(A)) \leq 1 \), then \( S_{k,m}(\Gamma_0(t)) \) is direct sum of oldforms and newforms.
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\[
S_{k,m}(\Gamma_0(p)) \quad \text{and} \quad S_{k,m}(GL_2(A))
\]

Theorem (Bandini, V. - 2020)

We have that \( S_{k,m}(\Gamma_0(p)) = S_{k,m}^{p-new}(\Gamma_0(p)) \oplus S_{k,m}^{p-old}(\Gamma_0(p)) \) if and only if the map
\[
D := Id - P^{k-2m}(Tr_{t(1)}^p)^2
\]
is bijective on \( S_{k,m}(\Gamma_0(p)) \).
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---

**Theorem (Bandini, V. - 2020)**

If the map \( D := Id - (\pi P)^{k-2m}(Tr_{m}^{mp})^2 \) is bijective on \( S_{k,m}(\Gamma_0(mp)) \), then we have the direct sum decomposition
\[
S_{k,m}(\Gamma_0(mp)) = S_{k,m}^{p-new}(\Gamma_0(mp)) \oplus S_{k,m}^{p-old}(\Gamma_0(mp))
\]

---

**Problem:** to get the full equivalence \( Ker(D) \) should contain a form \( f \neq 0 \) and also \( Fr^{(m)}(f) \) for a suitable \( f \in S_{k,m}(\Gamma_0(m)) \).
Let \( n = (\nu), \mathfrak{d} = (\delta) \subset A \) be ideals such that \( \mathfrak{d} \parallel n \). Denote by \( W^n_\mathfrak{d} \) a matrix of the form
\[
\begin{pmatrix}
\delta a & b \\

\nu c & \delta d \\
\end{pmatrix}
\]
with \( a, b, c, d \in A, \delta^2 ad - \nu cb = \zeta \delta \) and \( \zeta \in \mathbb{F}_q^* \).

It is easy to verify that such matrices are in the normalizer of \( \Gamma_0(n) \).
Let \( n = (\nu), \mathfrak{d} = (\delta) \subset A \) be ideals such that \( \mathfrak{d} \| n \). Denote by \( W^n_{\mathfrak{d}} \) a matrix of the form
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**Definition**

Let \( n = (\nu), \mathfrak{d} = (\delta) \subset A \) be ideals such that \( \mathfrak{d} \| n \). The **(partial) Atkin–Lehner involution** \( W^n_{\mathfrak{d}} \) acting on \( S_{k,m}(\Gamma_0(n)) \) is:

\[
W^n_{\mathfrak{d}} : S_{k,m}(\Gamma_0(n)) \to S_{k,m}(\Gamma_0(n))
\]

\[
f(z) \mapsto (f|_{k,m} W^n_{\mathfrak{d}})(z)
\]

for any \( W^n_{\mathfrak{d}} \) as above.
Let \( n = (\nu), d = (\delta) \subset A \) be ideals such that \( d || n \). Denote by \( W^n_\delta \) a matrix of the form

\[
\begin{pmatrix}
\delta a & b \\
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\end{pmatrix}
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W^n_\delta : S_{k,m}(\Gamma_0(n)) \to S_{k,m}(\Gamma_0(n))
\]

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for any \( W^n_\delta \) as above.

If \( d = n \) we have that \( W^n_n \) is the (full) Atkin–Lehner involution (Fricke involution) and it can be represented by the matrix

\[
W^n_n = \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}.
\]
Lemma (V. - 2021)

Let \( n = (\nu), \mathfrak{d} = (\delta) \subset A \) be ideals such that \( \mathfrak{d} | n \). If \( f \in S_{k,m}(\Gamma_0(\mathfrak{d})) \) then

\[
f |_{k,m} \left( \begin{array}{cc} \nu & 0 \\ \delta & 1 \end{array} \right) := D_{\frac{n}{\mathfrak{d}}}(f) = f |_{k,m} \left( W_n^\mathfrak{d} \right) \in S_{k,m}(\Gamma_0(n)).
\]
Lemma (V. - 2021)

Let \( n = (\nu), \mathfrak{d} = (\delta) \subset A \) be ideals such that \( \mathfrak{d}||n \). If \( f \in S_{k,m}(\Gamma_0(\mathfrak{d})) \) then

\[
f |_{k,m} \begin{pmatrix} \frac{\nu}{\delta} & 0 \\ 0 & 1 \end{pmatrix} := D_{\frac{n}{\delta}}(f) = f |_{k,m} W_{\frac{n}{\delta}}^n \in S_{k,m}(\Gamma_0(n)).
\]

○ If \( m = (\pi), p = (P) \subset A \) with \( (\pi, P) = 1 \) and \( P \) irreducible
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\]

○ If \( m = (\pi) \), \( p = (P) \subset A \) with \( (\pi, P) = 1 \) and \( P \) irreducible

\[
S_{k,m}(\Gamma_0(mp)) \\
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\[
D_p(f) = W_{mp}(f) \\
D_1(f) = W_{mp}(f)
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Lemma (V. - 2021)

Let \( n = (\nu), d = (\delta) \subset A \) be ideals such that \( d \| n \). If \( f \in S_{k,m}(\Gamma_0(d)) \) then

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○ If \( m = (\pi), p = (P) \subset A \) with \( (\pi, P) = 1 \) and \( P \) irreducible

\[
\begin{align*}
S_{k,m}(\Gamma_0(mp)) & \xrightarrow{D_p(f), D_1(f)} S_{k,m}(\Gamma_0(m)) \ni f \\
D_p(f) & = W_{mp}^p(f) \\
D_1(f) & = W_{mp}^1(f)
\end{align*}
\]
Lemma (V. - 2021)

Let $n = (\nu), \mathfrak{d} = (\delta) \subset A$ be ideals such that $\mathfrak{d} | n$. If $f \in S_{k,m}(\Gamma_0(\mathfrak{d}))$ then

$$f |_{k,m}\left(\begin{array}{cc}
\frac{\nu}{\delta} & 0 \\
0 & 1
\end{array}\right) := D_{\frac{n}{\delta}}(f) = f |_{k,m} W_{\mathfrak{n}}^{\mathfrak{n}} \in S_{k,m}(\Gamma_0(n)).$$

○ If $m = (\pi), p = (P) \subset A$ with $(\pi, P) = 1$ and $P$ irreducible

$$S_{k,m}(\Gamma_0(mp)) \ni D_p(f), D_1(f)$$

$$S_{k,m}(\Gamma_0(m)) \ni f$$

$$D_p(f) = W_p^{mp}(f)$$

$$D_1(f) = W_1^{mp}(f)$$

$$S_{k,m}^{p-old}(\Gamma_0(mp)) = \text{Span}\{W_1^{mp}(S_{k,m}(\Gamma_0(m))), W_p^{mp}(S_{k,m}(\Gamma_0(m)))\}.$$
Lemma (V. - 2021)

Let $\mathfrak{n} = (\nu), \mathfrak{d} = (\delta) \subset A$ be ideals such that $\mathfrak{d} \| \mathfrak{n}$. If $f \in S_{k,m}(\Gamma_0(\mathfrak{d}))$ then

$$
\begin{pmatrix}
\nu & 0 \\
\delta & 1
\end{pmatrix} := D_{\mathfrak{n} \delta}(f) = f \mid_{k,m} W_{\mathfrak{n} \delta} \in S_{k,m}(\Gamma_0(\mathfrak{n})).
$$

○ If $\mathfrak{m} = (\pi), \mathfrak{p} = (P) \subset A$ with $(\pi, P) = 1$ and $P$ irreducible

\begin{align*}
S_{k,m}(\Gamma_0(\mathfrak{mp})) &\ni D_p(f), D_1(f) \\
S_{k,m}(\Gamma_0(\mathfrak{m})) &\ni f
\end{align*}

$$
\begin{align*}
D_p(f) &= W_{\mathfrak{mp}}(f) \\
D_1(f) &= W_{1\mathfrak{mp}}(f)
\end{align*}
$$

$$
S_{k,m}^{p-old}(\Gamma_0(\mathfrak{mp})) = Span\{W_{1\mathfrak{mp}}(S_{k,m}(\Gamma_0(\mathfrak{m}))), W_{\mathfrak{mp}}(S_{k,m}(\Gamma_0(\mathfrak{m})))\}.
$$

Theorem (V. - 2021)

With assumptions on $\mathfrak{m}$ and $\mathfrak{p}$ as above, let $\mathfrak{d} = (\delta)$ be such that $\delta \| \pi$. Then

$$
\begin{align*}
W_{\delta \mathfrak{mp}}(T_p(f)) &= T_p(W_{\delta \mathfrak{mp}}(f)) \text{ if } f \in S_{k,m}(\Gamma_0(\mathfrak{m})) \\
W_{\delta \mathfrak{mp}}(U_p(f)) &= U_p(W_{\delta \mathfrak{mp}}(f)) \text{ if } f \in S_{k,m}(\Gamma_0(\mathfrak{mp}))
\end{align*}
$$
Recall that for $f \in S_{k,m}(\Gamma_0(mp))$, the trace is $\text{Tr}_m^{mp}(f) = \sum_{\gamma \in R_m^{mp}} f|_{k,m \gamma}$. 

**Proposition (V. - 2021)**

With notations as above, we have:

$$W_{mp}^{mp} \circ \text{Tr}_m^{mp}(d) = \delta_2^{mp} - k \text{Tr}_m^{mp}(mp^{2}d)$$

if $p \divides d$.

The above proposition implies

$$\text{Ker}(\text{Tr}_m^{mp}(mp)) = \text{Ker}(\text{Tr}_m^{mp}(p)).$$
Recall that for \( f \in S_{k,m}(\Gamma_0(mp)) \), the trace is \( Tr^mp(f) = \sum_{\gamma \in R^mp} f|_{k,m\gamma} \).

**Definition**

For a \( f \in S_{k,m}(\Gamma_0(mp)) \) and any divisor \( \mathfrak{d} \) of \( mp \) such that \( \mathfrak{d} | mp \), we define the \( \mathfrak{d} \)-twisted trace map as

\[
Tr^mp(\mathfrak{d}) := Tr^mp \circ W^mp_{\mathfrak{d}} : S_{k,m}(\Gamma_0(mp)) \to S_{k,m}(\Gamma_0(m))
\]

\[
f \mapsto \sum_{\gamma \in R^mp_{\mathfrak{d}}} (f|_{k,mW^mp_{\mathfrak{d}}})|_{k,m\gamma}.
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Recall that for \( f \in S_{k,m}(\Gamma_0(mp)) \), the trace is \( Tr_m^{mp}(f) = \sum_{\gamma \in R_m^{mp}} f_{k,m}\gamma \).

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\[
Tr_m^{mp(\mathfrak{d})} := Tr_m^{mp} \circ W_{\mathfrak{d}}^{mp} : S_{k,m}(\Gamma_0(mp)) \to S_{k,m}(\Gamma_0(m))
\]

\[
f \mapsto \sum_{\gamma \in R_m^{mp}} (f_{k,m} W_{\mathfrak{d}}^{mp})_{k,m}\gamma.
\]

**Proposition (V. - 2021)**

With notations as above, we have:

\[
W_{\mathfrak{m}}^{mp} \circ Tr_m^{mp(\mathfrak{d})} = \begin{cases} 
\delta^{2m-k} Tr_m^{mp(\frac{m}{\mathfrak{d}})} & \text{if } p \nmid \mathfrak{d} \\
(\frac{\delta}{p})^{2m-k} Tr_m^{mp(\frac{mp^2}{\mathfrak{d}})} & \text{if } p|\mathfrak{d}
\end{cases}
\]
○ Recall that for $f \in S_{k,m}(\Gamma_0(mp))$, the trace is $\text{Tr}_m^{mp}(f) = \sum_{\gamma \in R_m^{mp}} f|_{k,m\gamma}$.

**Definition**

For a $f \in S_{k,m}(\Gamma_0(mp))$ and any divisor $\varnothing$ of $mp$ such that $\varnothing||mp$, we define the $\varnothing$-twisted trace map as

$$\text{Tr}_m^{mp}(\varnothing) := \text{Tr}_m^{mp} \circ \text{W}_{\varnothing}^{mp} : S_{k,m}(\Gamma_0(mp)) \to S_{k,m}(\Gamma_0(m))$$

$$f \mapsto \sum_{\gamma \in R_m^{mp}} (f|_{k,m\gamma} W_{\varnothing}^{mp})|_{k,m\gamma}.$$ 

**Proposition (V. - 2021)**

With notations as above, we have:

$$\text{W}_{m}^{mp} \circ \text{Tr}_m^{mp}(\varnothing) = \begin{cases} 
\delta^{2m-k} \text{Tr}_m^{mp}(\frac{m}{\varnothing}) & \text{if } p \nmid \varnothing \\
(\frac{\varnothing}{p})^{2m-k} \text{Tr}_m^{mp}(\frac{mp^2}{\varnothing}) & \text{if } p|\varnothing.
\end{cases}$$

The above proposition implies

$$\text{Ker}(\text{Tr}_m^{mp}(mp)) = \text{Ker}(\text{Tr}_m^{mp}(p)).$$
Corollary

The space of \(p\)-newforms of level \(mp\), denoted by \(S_{k,m}^{p-new}(\Gamma_0(mp))\), is given by \(\text{Ker}(\mathbf{T}r_{mp}) \cap \text{Ker}(\mathbf{T}r_{mp}(p))\).
Corollary

The space of $p$-newforms of level $mp$, denoted by $S_{k,m}^{p-new}(\Gamma_0(mp))$, is given by

$$Ker(Tr_m^{mp}) \cap Ker(Tr_m^{mp(p)}).$$

Theorem (V. - 2021)

The map $D := Id - P^{k-2m}(Tr_m^{mp(p)})^2$ is bijective on $S_{k,m}(\Gamma_0(mp))$ if and only if we have the direct sum decomposition $S_{k,m}(\Gamma_0(mp)) = S_{k,m}^{p-new}(\Gamma_0(mp)) \oplus S_{k,m}^{p-old}(\Gamma_0(mp))$. 
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**Action on cusps**

$$\begin{align*}
\text{Level } mp \\
\left( \begin{array}{c}
1 \\
1 \\
\end{array} \right) \\
W_{mp}^{mp} \\
\left( \begin{array}{c}
1 \\
\pi P \\
\end{array} \right)
\end{align*}$$
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The space of $p$-newforms of level $mp$, denoted by $S_{k,m}^{p-new}(\Gamma_0(mp))$, is given by $\text{Ker}(Tr_m^{mp}) \cap \text{Ker}(Tr_m^{mp(p)})$.

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**Action on cusps**

- **Level $mp$**
  - $(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})$
  - $(\begin{smallmatrix} 1 P \\ \pi \end{smallmatrix})$

- **Level $m$**
  - $(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix})$
  - $(\begin{smallmatrix} 1 \pi \\ \pi \end{smallmatrix})$
Corollary

The space of $p$-newforms of level $mp$, denoted by $S_{k,m}^{p-new}(\Gamma_0(mp))$, is given by $Ker(Tr_{m}^{mp}) \cap Ker(Tr_{m}^{mp(p)})$.

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Action on cusps

\[
\begin{align*}
\text{Level } mp & \quad \text{Level } m & \quad \text{Level } mp \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
W_{mp}^{mp} & \quad W_{mp}^{mp} & \quad W_{mp}^{mp} \\
\begin{pmatrix} 1 \\ \pi P \end{pmatrix} & \quad \begin{pmatrix} 1 \\ \pi \end{pmatrix} & \quad \begin{pmatrix} 1 \\ P \end{pmatrix}
\end{align*}
\]
Corollary

The space of $p$-newforms of level $mp$, denoted by $S_{k,m}^{p-new}(\Gamma_0(mp))$, is given by $\text{Ker}(Tr_m^{mp}) \cap \text{Ker}(Tr_m^{mp(p)})$.

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Action on cusps
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The space of $p$-newforms of level $mp$, denoted by $S_{k,m}^{p-new}(\Gamma_0(mp))$, is given by $Ker(T_{mp}^m) \cap Ker(T_{mp}^{m(p)})$.

Theorem (V. - 2021)

The map $D := Id - P^{k-2m}(T_{mp}^{m(p)})^2$ is bijective on $S_{k,m}(\Gamma_0(mp))$ if and only if we have the direct sum decomposition $S_{k,m}(\Gamma_0(mp)) = S_{k,m}^{p-new}(\Gamma_0(mp)) \oplus S_{k,m}^{p-old}(\Gamma_0(mp))$.

Action on cusps

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</tr>
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Proposition (V. - 2020)

The involution $W_{mp}^{mp}$ and the operator $U_p$ commute on the space of $p$-newforms of level $mp$. 
Lemma (V. - 2021)

Let $f \in S_{k,m}(\Gamma_0(p))$ be a $p$-newform of level $p$. Then, $D_1(f), D_m(f) \in S_{k,m}(\Gamma_0(mp))$ are $p$-newforms of level $mp$. 
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\[
\begin{align*}
\circ f &= \sum_{i \geq 0} a_i u(z)^i, \quad a_i \in A. \\
\circ v_p(f) &= \inf_i v_p(a_i).
\end{align*}
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- \( f = \sum_{i \geq 0} a_i u(z)^i, \ a_i \in A. \)
- \( v_p(f) = \inf_i v_p(a_i). \)
- We say that \( f \) is a \( p \)-adic Drinfeld modular form for \( \Gamma_0(m) \) if it exists a sequence \( \{f_i\} \) of Drinfeld modular forms for \( \Gamma_0(m) \) verifying \( v_p(f_i - f) \to \infty \) as \( i \to \infty \).


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Thanks for your attention!