Bounded functional calculi for unbounded operators

Charles Batty (University of Oxford)

with Alexander Gomilko and Yuri Tomilov

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What is a bounded functional calculus for operators?

Operator A on a Banach space X, a class $\mathcal{F}$ of functions defined on the spectrum $\sigma(A)$. A functional calculus for A is an assignment

$$f \in \mathcal{F} \mapsto f(A) \quad \text{(operator on } X\text{)}$$

which reflects the structure of $\mathcal{F}$ and relates sensibly to A.

Often $\mathcal{F}$ is a (Banach) algebra, and this assignment is a (bounded) algebra homomorphism of $\mathcal{F}$ into $L(X)$. If $-z \in \rho(A)$ and $r_z(w) = (z+w)^{-1}$, then $r_z(A)$ should be $(z+A)^{-1}$, and so on. In this situation we have a *bounded functional calculus*.

If X is a Hilbert space and A is self-adjoint, there is a bounded functional calculus for bounded measurable functions on $\sigma(A)$.

Otherwise, the functions will normally be holomorphic on a set containing $\sigma(A)$, for example the Riesz-Dunford calculus

$$f(A) = \frac{1}{2\pi i} \int_\gamma f(z)(z-A)^{-1}\, dz.$$

Here A may be bounded, or unbounded with special properties.

Sectorial functional calculus

If A is an (injective) sectorial operator of angle $\theta \in (0, \pi)$, one can define a functional calculus for bounded holomorphic functions on sectors Σ_ψ, where $\psi \in (\theta, \pi)$, based on a Riesz-Dunford integral and an extension process (McIntosh, Haase, etc).

This produces a bounded functional calculus for many differential operators, but not for all sectorial operators.

There is a similar procedure for half-planes.

Semigroup generators and Hille-Phillips calculus

Let $-A$ be the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$. on X, and $K = \sup_{t\geq 0} \|T(t)\|$. Then, in some sense,

$$T(t) = e_t(A), \qquad e_t(z) = e^{-tz}, \quad z \in \mathbb{C}_+,$$

$$(z+A)^{-1}x = \int_0^\infty e^{-tz}T(t)x\,dt, \quad z \in \mathbb{C}_+, x \in X.$$

Let $\mathcal{LM} := \{\widehat{\mu} : \mu \in M(\mathbb{R}_+)\}$. Write m for $\widehat{\mu}$.

$$m(z) = \int_{\mathbb{R}_+} e^{-zt}\,d\mu(t) \quad (z \in \mathbb{C}_+),$$

$$m(A)x := \int_{\mathbb{R}_+} T(t)x\,d\mu(t),$$

$$\|m(A)\| \leq K\|\mu\|_{M(\mathbb{R}_+)} =: K\|m\|_{\mathrm{HP}}.$$

The norm-estimate is often far from sharp.

Cayley transform question

Suppose that $-A$ generates a bounded C_0-semigroup. The operator $V(A) := (A - I)(A + I)^{-1}$ is the cogenerator. The question was raised whether $V(A)$ is power-bounded. One approach is as follows.

Let

$$f_n(z) = \left(\frac{z-1}{z+1}\right)^n .$$

Then $f_n \in \mathcal{LM}$, $f_1 = \hat{\mu}$, $d\mu = \delta_0 - 2e^{-t}dt$, $\|f_1\|_{\mathrm{HP}} = 3$,

$$\|f_n\|_{\mathrm{HP}} \asymp n^{1/2}.$$

So this tells us that $\|V(A)^n\|$ do not grow faster than $n^{1/2}$. Can we do better?

An alternative formula

Assume that $\mu(\{0\}) = 0$. Informally,

$$\begin{aligned}
m(A)x &= \int_0^\infty T(t)x\,d\mu(t) \\
&= 4\int_0^\infty \int_0^\infty \tau t^2 e^{-2\tau t} T(t)x\,d\tau\,d\mu(t) \\
&= \frac{2}{\pi}\int_{\mathbb{R}} \tau \int_0^\infty \tau t e^{-\tau t} \int_{\mathbb{R}} (\tau - is + A)^{-2} x\, e^{-ist}\,ds\,d\tau\,d\mu(t) \\
&= \frac{2}{\pi}\int_{\mathbb{R}} \int_0^\infty \tau(\tau - is + A)^{-2} x \int_0^\infty t e^{-(\tau+is)t}\,d\mu(t)\,d\tau\,ds \\
&= -\frac{2}{\pi}\int_{\mathbb{R}} \int_0^\infty \tau(\tau - is + A)^{-2} x\, m'(\tau + is)\,d\tau\,ds
\end{aligned}$$

A definition

Operator A with dense domain and $\sigma(A) \subseteq \overline{\mathbb{C}}_+$,

$f : \mathbb{C}_+ \to \mathbb{C}$ holomorphic, $f(\infty) = \lim_{t\to\infty} f(t)$, $x \in X, x^* \in X^*$

$$\langle f(A)x, x^* \rangle := f(\infty)\langle x, x^* \rangle - \frac{2}{\pi} \int_{\mathbb{R}} \int_0^\infty \tau \langle (\tau - is + A)^{-2} x, x^* \rangle f'(\tau + is) \, d\tau \, ds$$

This double integral is absolutely convergent if $f \in \mathcal{B} := B^0_{\infty,1}(\mathbb{C}_+)$, i.e., f is holomorphic on $\mathbb{C}_+$ and

$$\int_0^\infty \sup_{s\in\mathbb{R}} |f'(\tau + is)| \, d\tau < \infty,$$

and the functions $z \mapsto \langle (z + A)^{-1} x, x^* \rangle$ all belong to the space $\mathcal{E} := B^0_{1,\infty}(\mathbb{C}_+)$ of all holomorphic functions g on $\mathbb{C}_+$ such that

$$\sup_{\tau>0} \tau \int_{\mathbb{R}} |g'(\tau + is)| \, ds < \infty$$

Gomilko–Shi–Feng

Let A be a densely defined operator, $\sigma(A) \subseteq \overline{\mathbb{C}}_+$,
$z \mapsto \langle (z+A)^{-1}x, x^* \rangle \in \mathcal{E}$ for all $x \in X$, $x^* \in X^*$.
By the Closed Graph Theorem, there exists Γ_A such that

$$\tau \int_{\mathbb{R}} |\langle (\tau + is + A)^{-2}x, x^* \rangle| \, ds \le \Gamma_A \|x\| \, \|x^*\|. \tag{GSF}$$

1999/2000: Gomilko, and independently Shi and Feng, showed that (GSF) implies that $-A$ generates a bounded C_0-semigroup.

If $-A$ is the generator of a bounded C_0-semigroup on a Hilbert space, then (GSF) holds.

If A is sectorial of angle less than $\pi/2$, then (GSF) holds (and $-A$ generates a bounded holomorphic semigroup).

Let A be the generator of the C_0-group of shifts on $L^p(\mathbb{R})$, where $1 \le p < \infty$, $p \ne 2$. Then $\pm A$ do not satisfy (GSF).

The analytic Besov algebra $B^0_{\infty,1}(\mathbb{C}_+)$

Any $f \in \mathcal{B}$ extends to a bounded uniformly continuous function on $\overline{\mathbb{C}}_+$, and moreover $f(\infty) := \lim_{\tau\to\infty} f(\tau)$ exists. Furthermore, $\mathcal{B}$ is a Banach algebra in the norm

$$\|f\|_{\mathcal{B}} := \|f\|_\infty + \int_0^\infty \sup_{s\in\mathbb{R}} |f'(\tau + is)| \, d\tau.$$

There is a partial duality between $\mathcal{B}$ and $\mathcal{E}$:

$$\langle g, f\rangle_{\mathcal{B}} = \int_{\mathbb{R}} \int_0^\infty g'(\tau - is) f'(\tau + is) \, d\tau \, ds \qquad (g \in \mathcal{E}, f \in \mathcal{B})$$

Moreover

$$f(z) = f(\infty) + \frac{2}{\pi} \langle r_z, f\rangle_{\mathcal{B}}, \quad r_z(w) = (w+z)^{-1}.$$

Examples

1. *Hille-Phillips algebra*. Let $\mathcal{LM} := \{\widehat{\mu} : \mu \in M(\mathbb{R}_+)\}$. If $m \in \mathcal{LM}$, then $m \in \mathcal{B}$ and $\|m\|_{\mathcal{B}} \le 2\|m\|_{\mathrm{HP}}$.

2. *Entire functions of exponential type*. For $0 < \tau_1 < \tau_2 < \infty$, let $H^\infty[\tau_1, \tau_2]$ be the space of all $f \in H^\infty(\mathbb{C}_+)$ such that the "spectrum" of $f(i\cdot)$ is contained in $[\tau_1, \tau_2]$. Then f is an entire function of exponential type.

$H^\infty[\tau_1, \tau_2] \subset \mathcal{B}$ and

$$\|f\|_{\mathcal{B}} \le \left(1 + 4\log\left(1 + \frac{\tau_2}{\tau_1}\right)\right)\|f\|_\infty$$

(earlier partial results by White, Vitse, Haase, following a discrete version by Peller)

The closure of $\bigcup_{0<\tau_1<\tau_2} H^\infty[\tau_1, \tau_2]$ in $\mathcal{B}$ is $\{f \in \mathcal{B} : f(\infty) = 0\}$.

More examples

3. *Cayley transforms.* $f_n(z) := \left(\frac{z-1}{z+1}\right)^n$ is in $\mathcal{LM}$ and hence in $\mathcal{B}$. Moreover,

$$\|f_n\|_{\mathcal{B}} \asymp \log n, \qquad \|f_n\|_{\mathrm{HP}} \asymp n^{1/2}.$$

4. $e^{-1/z}$ is not in $\mathcal{B}$—it is not uniformly continuous near 0

$e^{-1/(z+1)}$ is in $\mathcal{B}$—in fact it is in $\mathcal{LM}$

$\left(\frac{z}{z+1}\right)^2 e^{-1/z}$ is also in $\mathcal{B}$; it is in the norm-closure of $\mathcal{LM}$

5. If f is a Bernstein function, then $\left(\lambda + f(z^\alpha)^\beta\right)^{-1}$ is in $\mathcal{B}$, for $\alpha \in (0,1)$ and $\beta \in (1, 1/\alpha)$, $\lambda \in \mathbb{C}_+$.

Suppose that A satisfies (GSF). For $f \in \mathcal{B}$, define

$$\begin{aligned}\langle f(A)x, x^* \rangle &:= f(\infty)\langle x, x^* \rangle \\ &\qquad - \frac{2}{\pi} \int_{\mathbb{R}} \int_0^\infty \tau \langle (\tau - is + A)^{-2} x, x^* \rangle f'(\tau + is) \, d\tau \, ds\end{aligned}$$

Then $f(A)x \in X^{**}$, $f(A) : X \to X^{**}$, $\|f(A)\| \leq 2\Gamma_A \|f\|_{\mathcal{B}}$.

Does $f(A)$ map X into X?

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Yes, if $f \in \mathcal{LM}$; moreover, $f(A)$ agrees with the Hille-Phillips functional calculus, i.e., $f(A)x = \int_0^\infty e^{-tA}x \, d\mu(t)$ if $f(z) = \int_0^\infty e^{-tz} \, d\mu(t)$.

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Yes, if $f \in H^\infty[\tau_1, \tau_2]$ where $0 < \tau_1 < \tau_2 < \infty$. Let $\delta > 0$,

$$e_\delta(z) = \frac{1 - e^{-\delta z}}{\delta z}$$

Then $e_\delta \in \mathcal{LM}$, $fe_\delta \in \mathcal{LM}$, and $\lim_{\delta \to 0+}(fe_\delta)(A) = f(A)$ in the strong operator topology.

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Yes, for all $f \in \mathcal{B}$ because $f - f(\infty)$ is in the norm-closure of $\bigcup_{0<\tau_1<\tau_2} H^\infty[\tau_1, \tau_2]$.

So $f \mapsto f(A)$ is a bounded algebra homomorphism from $\mathcal{B}$ to $L(X)$, extending the Hille-Phillips calculus.

The $\mathcal{B}$-calculus

If A is an operator such that $\sigma(A) \subseteq \overline{\mathbb{C}}_+$ and there is a bounded algebra homomorphism $\Phi : \mathcal{B} \to L(X)$ such that

$$\Phi(r_\lambda) = (\lambda + A)^{-1}, \qquad r_\lambda(z) = (\lambda + z)^{-1}, \quad \lambda, z \in \mathbb{C}_+.$$

Then

- A satisfies (GSF),
- Φ is the homomorphism $f \mapsto f(A)$ defined above.

We call this the $\mathcal{B}$-calculus (or the Besov calculus).

The $\mathcal{B}$-calculus is compatible with:

- Sectorial calculus
- Half-plane calculus

in the sense that the $\mathcal{B}$-calculus definition of $f(A)$ agrees with definitions of $f(A)$ in these calculi whenever they can also be defined.

Convergence Lemma

Theorem

Let A be an operator satisfying (GSF). Let (f_n) be a sequence in $\mathcal{B}$ with $\sup_n \|f_n\|_{\mathcal{B}} < \infty$, and assume that

- $f(z) := \lim_{n\to\infty} f_n(z)$ *exists for all* $z \in \mathbb{C}_+$,
- *For all* $r > 0$,

$$\lim_{\delta\to 0+} \int_0^\delta \sup_{|\beta|\le r} |f_n'(\alpha + i\beta)|\, d\alpha = 0,$$

  uniformly in n.

Then $f \in \mathcal{B}$ *and* $\lim_{n\to\infty} f_n(A) = f(A)$ *in the strong operator topology.*

Spectral Inclusion

Theorem

Let $f \in \mathcal{B}$.

1. *If A satisfies (GSF), then $f(\sigma(A)) \subseteq \sigma(f(A))$.*
2. *If A is sectorial of angle less than $\pi/2$ then* $f(\sigma(A)) \cup \{f(\infty)\} = \sigma(f(A)) \cup \{f(\infty)\}$.

Cayley transforms

Theorem (Gomilko 2004)

Assume that $-A$ generates a bounded C_0-semigroup on a Hilbert space, with $\|e^{-tA}\| \le M$. Let $V(A) = (A-1)(A+1)^{-1}$. Then

$$\|V(A)^n\| \le cM^2(1 + \log n).$$

Inverse generator problem

$e^{-1/z}$ is not in $\mathcal{B}$—it is not uniformly continuous near 0

$e^{-1/(z+1)}$ is in $\mathcal{B}$—in fact it is in $\mathcal{LM}$

$\left(\frac{z}{z+1}\right)^2 e^{-1/z}$ is also in $\mathcal{B}$; it is in $\mathcal{LM}$

Theorem (*Zwart 2007)

Let $-A$ be the generator of a bounded C_0-semigroup on a Hilbert space with $\|e^{-tA}\| \le M$, and assume that A has a bounded inverse. Then

$$\|e^{-tA^{-1}}\| \le cM^2 \|(1+A^{-1})^2\| (1+\log(1+t)).$$

*Zwart assumed that $-A$ generates an exponentially stable C_0-semigroup.

Bernstein functions preserve sectoriality

Theorem (Gomilko–Tomilov 2015)

Let A be sectorial of angle $\omega \in [0, \pi/2)$, and let f be a Bernstein function. Then $f(A)$ is sectorial of angle ω (or less).

Bounded holomorphic semigroups

Suppose that A is sectorial of angle less than $\pi/2$, so that $\|z(z+A)^{-1}\| \le M_A$ for $z \in \mathbb{C}_+, z \neq 0$.

There is an absolute constant C such that

$$\tau \int_{\mathbb{R}} \|(\tau + A)^{-2}\| \, d\tau \le C M_A(\log M_A + 1) \qquad (\tau > 0)$$

Hence

$$\|f(A)\| \le C M_A (\log M_A + 1) \|f\|_{\mathcal{B}} \qquad (f \in \mathcal{B}).$$

This was first proved by Vitse, using dyadic decompositions, and with a somewhat larger "constant".

$\mathcal{D}$-calculus

$s > -1$, $f : \mathbb{C}_+ \to \mathbb{C}$ holomorphic; $f \in \mathcal{D}_s$ if

$$\|f\|_{\mathcal{D}_{s,0}} = \int_0^\infty \int_{\mathbb{R}} \frac{|f'(\alpha + i\beta)|}{(\alpha^2 + \beta^2)^{(s+1)/2}} \, d\beta \, d\alpha < \infty.$$

For A sectorial of angle less than $\pi/2$,

$$f_{\mathcal{D}}(A) := f(\infty) - \frac{2^s}{\pi} \int_0^\infty \int_{\mathbb{R}} f(\alpha + i\beta)(A + \alpha - i\beta)^{-(s+1)/2} \, d\beta \, d\alpha.$$

$\mathcal{B} \subset \mathcal{D}_s \subset \mathcal{D}_\tau \quad (0 < s < \tau)$

$\mathcal{D}_s$ is a Banach space, but it is not an algebra; $\bigcup_{s>0} \mathcal{D}_s$ is an algebra but not a Banach space; $\mathcal{D}_s \cap H^\infty(\mathbb{C}_+)$ is a Banach algebra.

$f_{\mathcal{D}}(A)$ does not depend on s, and it has the properties of a functional calculus.

Let $f_n(z) = \left(\frac{z-1}{z+1}\right)^n$. For $s > 0$, the sequence $\|f_n\|_{\mathcal{D}_s}$ is bounded.

Hence the Cayley transform question has a positive answer for bounded holomorphic C_0-semigroups (first proved by deLaubenfels in 1985)

If there is a $\mathcal{D}$-calculus for an operator A, then A is sectorial of angle less than $\pi/2$.

$\mathcal{H}$-calculus

$\psi \in (0, \pi)$, $f : \Sigma_\psi \to \mathbb{C}$ holomorphic; $f \in \mathcal{H}_\psi$ if $f' \in H^1(\Sigma_\psi)$

$\mathcal{H}_\psi$ is a Banach algebra in the norm

$$\|f\|_{\mathcal{H}_\psi} = \|f\|_{H^\infty(\Sigma_\psi)} + \|f'\|_{H^1(\Sigma_\psi)}.$$

A sectorial of angle $\theta \in (0, \pi/2)$, $\psi \in (\theta, \pi/2)$, $\gamma = \pi/(2\psi)$,

$f \in \mathcal{H}_\psi, g(z) = f(z^{1/\gamma})$. Then $g \in \mathcal{D}_s$ for all $s > -1$.

$$f_{\mathcal{H}}(A) := g_{\mathcal{D}}(A^\gamma).$$

This does not depend on ψ, and it defines a bounded functional calculus.

If there is a $\mathcal{H}_\psi$-calculus for an operator A, then A is sectorial of angle less than ψ.

ALM-calculus (Arnold–LeMerdy)

There is a Banach algebra $\mathcal{A}$ of holomorphic functions on $\mathbb{C}_+$ such that $\mathcal{B}$ is continuously included in $\mathcal{A}$, and every operator A which is the negative generator of a bounded C_0-semigroup on a Hilbert space has a bounded $\mathcal{A}$-calculus.

Question: Let $f_n(z) = \left(\frac{z-1}{z+1}\right)^n$. Is the sequence $\|f_n\|_{\mathcal{A}}$ bounded?

If the answer is Yes, then the Cayley transform question has a positive answer for bounded C_0-semigroups on Hilbert space.

Main references

$\mathcal{B}$-calculus

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$\mathcal{D}$-calculus and $\mathcal{H}$-calculus

Batty, Gomilko and Tomilov, arxiv.org:2101.05083

ALM-calculus

L. Arnold and C. Le Merdy, arxiv.org:2012.04440

CJKB

von Neumann's inequality

Let T be a contraction on Hilbert space, and $p(z)$ be a polynomial.

$$\|p(T)\| \leq \|p\|_\infty \qquad \text{(sup-norm on the unit disc)}$$

Then we can extend to the disc algebra, by continuity

$$\|f(T)\| \leq \|f\|_\infty$$

If $S = V^{-1}TV$,

$$f(S) = V^{-1}f(T)V$$

What about power-bounded operators in general?

Power-bounded operators on Hilbert space

Peller (1982):

T power-bounded on Hilbert space

Besov norm for holomorphic f on the unit disc $\mathbb{D}$:

$$\|f\|_B := \int_0^1 \sup_\theta |f'(re^{i\theta})| \, dr$$

$$\|p(T)\| \le C\|p\|_B$$

Hence there is a bounded functional calculus for the analytic Besov space $B^0_{\infty,1}(\mathbb{D})$ of functions $f \in H^\infty(\mathbb{D})$ for which $\|f\|_B < \infty$.

Applications

Theorem (Zwart 2012, Haase–Rozendaal 2013)

If $f \in H^\infty[\tau, \infty)$ and f extends to a function in H^∞_ω, and $-A$ generates a bounded C_0-semigroup on a Hilbert space, where $\|e^{-tA}\| \le M$, then

$$\|f(A)\| \le cM^2 e^{-\omega\tau} \|f\|_{H^\infty_\omega} \left(2 + \frac{1}{2}\log\left(1 + \frac{1}{\tau\omega}\right)\right).$$

Theorem (Schwenninger–Zwart*)

Assume that $-A$ *generates an exponentially stable* C_0*-semigroup on a Hilbert space, so that there exist* $M, \delta > 0$ *such that*

$$\|e^{-tA}\| \le Me^{-\delta t} \qquad (t \ge 0).$$

Let $f \in H^\infty(\mathbb{C}_+)$ *and assume that there is a monotonic decreasing function* $h : \mathbb{R}_+ \to (0, \infty)$ *such that*

$$|f(is)| \le h(|s|), \qquad \|h\|_\delta := \int_0^\infty \frac{h(t)}{t+\delta} < \infty.$$

Then

$$\|f(A)\| \le cM^2 \|h\|_\delta.$$

*Announced at Bedlewo, April 2017, for $h(s) = (\log(s+e))^{-\alpha}$ where $\alpha > 1$.

Appendix: proof of density

Proposition (Arveson?, Olesen?, Pedersen?)

Let $\{U(t) : t \in \mathbb{R}\}$ be a bounded C_0-group on a Banach space Y, with generator G, and let $Y(K)$ denote the spectral subspace corresponding to a closed subset K of $\mathbb{R}$. Assume that G has dense range. Then

$$\bigcup \{Y(K) : K \text{ compact}, 0 \notin K\}$$

is dense in Y.

Let $Y = \mathcal{B}_0 := \{f \in \mathcal{B} : f(\infty) = 0\}$. For $f \in \mathcal{B}_0$ and $a \in \overline{\mathbb{C}}_+$, let

$$(T_{\mathcal{B}}(a)f)(z) = f(z + a).$$

Then $\{T_{\mathcal{B}_0}(a) : a \in \mathbb{C}_+ \cup \{0\}\}$ is a holomorphic C_0-semigroup of contractions on $\mathcal{B}_0$, and $\{T_{\mathcal{B}_0}(is) : s \in \mathbb{R}\}$ is a C_0-group of isometries. Moreover the generator has dense range and its spectrum is $i\mathbb{R}_+$.

Littlewood-Paley decomposition

It follows that $\mathcal{B}$ is the space of all functions $f \in H^\infty(\mathbb{C}_+)$, such that the boundary function f^b has a Littlewood–Paley decomposition for $(0, \infty)$ of the form

$$f^b = f(\infty) + \sum_{k \in \mathbb{Z}} f^b * \psi_k,$$

which is absolutely convergent in the norm of $L^\infty(\mathbb{R})$.