Wigner's theorem in normed spaces

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Recent papers

- D. Ilišević and A. Turnšek, *On Wigner's theorem in strictly convex normed spaces*, Publ. Math. Debrecen 97 (2020), 393-401
- D. Ilišević and A. Turnšek, *On Wigner's theorem in smooth normed spaces*, Aequat. Math. 94 (2020), 1257-1267
- D. Ilišević, M. Omladič and A. Turnšek, *Phase-isometries between normed spaces*, Linear Algebra Appl. 612 (2021), 99-111
- D. Ilišević and A. Turnšek, *A variant of Wigner's theorem in normed spaces*, Mediterr. J. Math. 18, 148 (2021).

Theorem (Wigner)

Let $(H, (\cdot,\cdot))$ *and* $(K, (\cdot,\cdot))$ *be real or complex inner product spaces. A mapping* $f\colon H \to K$ *satisfies*

$$|(f(x), f(y))| = |(x, y)|, \quad x, y \in H,$$

if and only if there is a linear or an anti-linear isometry $U\colon H \to K$ *such that*

$$f(x) = \sigma(x)Ux, \quad x \in H,$$

where a so-called phase function σ *takes values in modulus one scalars; in other words* f *is phase equivalent to a linear or an anti-linear isometry.*

Semi-inner product

On each normed space X over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ there exists at least one semi-inner product on X which is a function $[\cdot,\cdot]\colon X \times X \to \mathbb{F}$ with the following properties:

1. $[x+y,z] = [x,z] + [y,z]$, $[\lambda x, y] = \lambda[x,y]$, $[x, \lambda y] = \overline{\lambda}[x,y]$ for all $\lambda \in \mathbb{F}$ and $x, y \in X$,
2. $[x,x] = \|x\|^2$ for all $x \in X$,
3. $|[x,y]| \leq \|x\|\|y\|$ for all $x, y \in X$.

However, in general, $[x,y]$ is not equal to $\overline{[y,x]}$ and $[x, y+z]$ is not equal to $[x,y] + [x,z]$.

Let X, Y be normed spaces and $f\colon X \to Y$ a mapping such that

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X.$$

Is it true that f satisfies the above equation if and only if it is phase equivalent to either a linear or an anti-linear isometry?

Example

Let $X = Y = \mathbb{R}^2$ with $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$, and let the semi-inner product for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be defined by

$$[x, y] = \begin{cases} x_1 y_1 & \text{if} \quad |y_1| > |y_2| \\ x_2 y_2 & \text{if} \quad |y_1| < |y_2| \\ \frac{3}{4} x_1 y_1 + \frac{1}{4} x_2 y_2 & \text{if} \quad |y_1| = |y_2|. \end{cases}$$

Define a surjective linear isometry $f\colon X \to Y$ by $f(x, y) = (y, x)$. For $x = (1, 0)$, $y = (1, 1)$ we have $[x, y] = \frac{3}{4}$ and $[f(x), f(y)] = \frac{1}{4}$.

Smooth normed spaces

Recall that X is said to have a Gateaux differentiable norm at $x \neq 0$ whenever

$$\lim_{t \to 0,\, t \in \mathbb{R}} \frac{\|x + ty\| - \|x\|}{t} \text{ exists for all } y \in X.$$

Recall also that a support functional ϕ_x at $x \in X$ is a norm-one linear functional in X^* such that $\phi_x(x) = \|x\|$.

A normed space X is said to be smooth at a nonzero $x \in X$ if there exists a unique support functional at x, and it is said to be **smooth** if it is smooth at each of its points.

It is well known that a Banach space X is smooth at x if and only if the norm is Gateaux differentiable at x. Moreover, in this case, the real part $\operatorname{Re} \phi_x$ of a unique support functional ϕ_x at x is given by

$$\operatorname{Re} \phi_x(y) = \lim_{t \to 0,\, t \in \mathbb{R}} \frac{\|x + ty\| - \|x\|}{t}.$$

If X is smooth, then $[x, y] := \|y\| \phi_y(x)$, where ϕ_y is the support functional at y, is the unique semi-inner product on X.

Isometries satisfy the generalized Wigner equation

Suppose that X and Y are smooth normed spaces. If U is a linear or an anti-linear isometry, then $\|Uy + tUx\| = \|y + tx\|$, $t \in \mathbb{R}$, hence

$$\operatorname{Re}\phi_{Uy}(Ux) = \operatorname{Re}\phi_y(x).$$

From this we conclude that $\phi_{Uy} \circ U = \phi_y$ if U is linear or $\phi_{Uy} \circ U = \overline{\phi_y}$ if U is anti-linear. In both cases we get $|[Ux, Uy]| = |[x, y]|$ for all $x, y \in X$. Then

$$|[f(x), f(y)]| = |[\sigma(x)Ux, \sigma(y)Uy]| = |[Ux, Uy]| = |[x, y]|.$$

Note that

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X$$

is equivalent to

$$|\phi_{f(y)}(f(x))| = |\phi_y(x)|, \quad x, y \in X.$$

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$$|\phi_{f(y)}(f(x))| = |\phi_y(x)|, \quad x, y \in X.$$

Theorem

Let X and Y be smooth normed spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and suppose that $f: X \to Y$ is a surjective mapping satisfying

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X.$$

Then the following holds.

(i) *If $\dim X \geq 2$ and $\mathbb{F} = \mathbb{R}$, then f is phase equivalent to a linear surjective isometry.*

(ii) *If $\dim X \geq 2$ and $\mathbb{F} = \mathbb{C}$, then f is phase equivalent to a linear or an anti-linear surjective isometry.*

Can we omit the surjectivity?

Example

Let $X = \mathbb{R}$ with the usual norm. Then X is smooth and the unique semi-inner product is given by $[x, y] = xy$, $x, y \in \mathbb{R}$.
Let $Y = \mathbb{R}^2$ with the max norm and let $f : X \to Y$ be given by

$$f(x) = (x, \sin x), \qquad x \in \mathbb{R}.$$

Since $f(y)$ is smooth for all $y \in X$, the support functional at $f(y)$ is $\phi_{f(y)}(z) = \frac{y}{|y|} z_1$, $z = (z_1, z_2) \in Y$.
Hence for any semi-inner product on Y we have

$$[f(x), f(y)] = \|f(y)\| \phi_{f(y)}(f(x)) = |y| \frac{y}{|y|} x = xy, \quad x, y \in X.$$

Therefore, $[f(x), f(y)] = [x, y]$, $x, y \in X$, but f is not phase equivalent to any linear isometry.

Example

Let $X = Y = \mathbb{R}^2$ with the max norm, that is, $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$, and let the semi-inner product for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be defined by

$$[x, y] = \begin{cases} x_1 y_1 & \text{if} \quad |y_1| \geq |y_2| \\ x_2 y_2 & \text{if} \quad |y_1| < |y_2|. \end{cases}$$

Define $f\colon X \to Y$ by $f((\lambda, \lambda)) = (\lambda, -\lambda)$, $f((\lambda, -\lambda)) = (\lambda, \lambda)$ and $f(x) = x$ for all other directions.
Then f is surjective and $|[f(x), f(y)]| = |[x, y]|$, $x, y \in X$.
However, f is not phase equivalent to any linear isometry from X to Y.

Theorem

Let X and Y be normed spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : X \to Y$ a surjective mapping. Suppose that for all semi-inner products on X and Y we have

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X.$$

Then the following holds.

- **(i)** *If* $\dim X = 1$, *then f is phase equivalent to a linear surjective isometry.*
- **(ii)** *If* $\dim X \geq 2$ *and* $\mathbb{F} = \mathbb{R}$, *then f is phase equivalent to a linear surjective isometry.*
- **(iii)** *If* $\dim X \geq 2$ *and* $\mathbb{F} = \mathbb{C}$, *then f is phase equivalent to a linear or an anti-linear surjective isometry.*

To prove that for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$ there exist $\omega_1, \omega_2 \in \mathbb{T} = \{t \in \mathbb{F} : |t| = 1\}$ such that $f(\alpha x + \beta y) = \omega_1 \alpha f(x) + \omega_2 \beta f(y)$.

Birkhoff–James orthogonality: $x \perp y \overset{def}{\Leftrightarrow} \|x + \lambda y\| \geq \|x\| \ \forall \lambda \in \mathbb{F}$.

Theorem (Fundamental theorem of projective geometry)

Let X and Y be vector spaces over $\mathbb{F}$ of dimensions at least three. Let $g\colon \{\langle x \rangle : x \in X\} \to \{\langle y \rangle : y \in Y\}$ be a mapping such that

(i) *The image of g is not contained in a two-dimensional subspace of Y.*

(ii) $0 \neq c \in \langle a, b \rangle, a \neq 0 \neq b$, *implies* $g(\langle c \rangle) \in \langle g(\langle a \rangle), g(\langle b \rangle) \rangle$.

Then there exists an injective semilinear mapping $A\colon X \to Y$ (that is, A is additive and $A(\lambda x) = h(\lambda) Ax$ for all $x, y \in X$ and $\lambda \in \mathbb{F}$, where $h\colon \mathbb{F} \to \mathbb{F}$ is a homomorphism) such that

$$g(\langle x \rangle) = \langle Ax \rangle, \quad 0 \neq x \in X.$$

Moreover, A is unique up to a non-zero scalar factor.

Let X and Y be real normed spaces. If $f\colon X \to Y$ is phase equivalent to a linear isometry, then f satisfies

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\}, \quad x, y \in H.$$

Example

$f\colon \mathbb{R} \to \mathbb{R}^2$ (with max norm), $f(x) = (x, \sin x)$ satisfies

$$\{\|f(x)+f(y)\|, \|f(x)-f(y)\|\} = \{\|x+y\|, \|x-y\|\}, \quad x, y \in \mathbb{R}.$$

but it is not phase equivalent to any linear isometry.

Recall that a normed space X is said to be **strictly convex** whenever the unit sphere S_X does not contain non-trivial line segments, that is, each point of S_X is an extreme point of the unit ball B_X.
In a strictly convex normed space the midpoint $z = \frac{x+y}{2}$ of the segment $[x, y]$ is characterized by $\|z - x\| = \|z - y\| = \frac{\|x-y\|}{2}$.

Proposition

Let X, Y be real smooth normed spaces, Y strictly convex, $f : X \to Y$ surjective. The following assertions are equivalent:

- **(i)** $|[f(x), f(y)]| = |[x, y]|$, $x, y \in X$;
- **(ii)** *f is phase equivalent to a surjective linear isometry;*
- **(iii)** $\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\}$, $x, y \in X$.

Theorem

Let X and Y be real normed spaces. Then a surjective mapping $f : X \to Y$ satisfies

$$\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\}, \quad x, y \in X,$$

if and only if f is phase equivalent to a surjective linear isometry.

Our approach to the problem is motivated by the proof of the Mazur–Ulam theorem on isometries between real normed spaces.

We show that the midpoint of the segment $[x, y]$ is mapped to one of the midpoints of the segments $[\pm f(x), \pm f(y)]$.