Wigner’s theorem in normed spaces

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Theorem (Wigner)

Let $(H, (\cdot, \cdot))$ and $(K, (\cdot, \cdot))$ be real or complex inner product spaces. A mapping $f : H \rightarrow K$ satisfies

$$|(f(x), f(y))| = |(x, y)|, \quad x, y \in H,$$

if and only if there is a linear or an anti-linear isometry $U : H \rightarrow K$ such that

$$f(x) = \sigma(x)Ux, \quad x \in H,$$

where a so-called phase function $\sigma$ takes values in modulus one scalars; in other words $f$ is phase equivalent to a linear or an anti-linear isometry.
On each normed space $X$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ there exists at least one semi-inner product on $X$ which is a function $[\cdot, \cdot] : X \times X \to \mathbb{F}$ with the following properties:

1. $[x + y, z] = [x, z] + [y, z]$, $[\lambda x, y] = \lambda [x, y]$, $[x, \lambda y] = \overline{\lambda} [x, y]$ for all $\lambda \in \mathbb{F}$ and $x, y \in X$,
2. $[x, x] = \|x\|^2$ for all $x \in X$,
3. $|[x, y]| \leq \|x\| \|y\|$ for all $x, y \in X$.

However, in general, $[x, y]$ is not equal to $[y, x]$ and $[x, y + z]$ is not equal to $[x, y] + [x, z]$. 
A generalized Wigner equation

Let $X, Y$ be normed spaces and $f : X \to Y$ a mapping such that

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X.$$ 

Is it true that $f$ satisfies the above equation if and only if it is phase equivalent to either a linear or an anti-linear isometry?

**Example**

Let $X = Y = \mathbb{R}^2$ with $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$, and let the semi-inner product for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be defined by

$$[x, y] = \begin{cases} 
  x_1y_1 & \text{if } |y_1| > |y_2| \\
  x_2y_2 & \text{if } |y_1| < |y_2| \\
  \frac{3}{4}x_1y_1 + \frac{1}{4}x_2y_2 & \text{if } |y_1| = |y_2|.
\end{cases}$$

Define a surjective linear isometry $f : X \to Y$ by $f(x, y) = (y, x)$. For $x = (1, 0), y = (1, 1)$ we have $[x, y] = \frac{3}{4}$ and $[f(x), f(y)] = \frac{1}{4}$. 

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Wigner’s theorem in normed spaces
Recall that $X$ is said to have a Gateaux differentiable norm at $x \neq 0$ whenever

$$
\lim_{t \to 0, \ t \in \mathbb{R}} \frac{\|x + ty\| - \|x\|}{t}
$$

exists for all $y \in X$.

Recall also that a support functional $\phi_x$ at $x \in X$ is a norm-one linear functional in $X^*$ such that $\phi_x(x) = \|x\|$.

A normed space $X$ is said to be smooth at a nonzero $x \in X$ if there exists a unique support functional at $x$, and it is said to be **smooth** if it is smooth at each of its points.

It is well known that a Banach space $X$ is smooth at $x$ if and only if the norm is Gateaux differentiable at $x$. Moreover, in this case, the real part $\text{Re} \phi_x$ of a unique support functional $\phi_x$ at $x$ is given by

$$
\text{Re} \phi_x(y) = \lim_{t \to 0, \ t \in \mathbb{R}} \frac{\|x + ty\| - \|x\|}{t}.
$$

If $X$ is smooth, then $[x, y] := \|y\| \phi_y(x)$, where $\phi_y$ is the support functional at $y$, is the unique semi-inner product on $X$. 
Isometries satisfy the generalized Wigner equation

Suppose that $X$ and $Y$ are smooth normed spaces. If $U$ is a linear or an anti-linear isometry, then $\|Uy + tUx\| = \|y + tx\|$, $t \in \mathbb{R}$, hence

$$\text{Re} \, \phi_{Uy}(Ux) = \text{Re} \, \phi_y(x).$$

From this we conclude that $\phi_{Uy} \circ U = \phi_y$ if $U$ is linear or $\phi_{Uy} \circ U = \overline{\phi_y}$ if $U$ is anti-linear. In both cases we get $|[Ux, Uy]| = |[x, y]|$ for all $x, y \in X$. Then

$$|[f(x), f(y)]| = |[\sigma(x)Ux, \sigma(y)Uy]| = |[Ux, Uy]| = |[x, y]|.$$

Note that

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X$$

is equivalent to

$$|\phi_{f(y)}(f(x))| = |\phi_y(x)|, \quad x, y \in X.$$
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$$|[f(x), f(y)]| = |[\sigma(x)Ux, \sigma(y)Uy]| = |[Ux, Uy]| = |[x, y]|.$$

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Theorem

Let $X$ and $Y$ be smooth normed spaces over $F \in \{\mathbb{R}, \mathbb{C}\}$ and suppose that $f : X \to Y$ is a surjective mapping satisfying

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X.$$ 

Then the following holds.

(i) If $\dim X \geq 2$ and $F = \mathbb{R}$, then $f$ is phase equivalent to a linear surjective isometry.

(ii) If $\dim X \geq 2$ and $F = \mathbb{C}$, then $f$ is phase equivalent to a linear or an anti-linear surjective isometry.
Can we omit the surjectivity?

Example

Let $X = \mathbb{R}$ with the usual norm. Then $X$ is smooth and the unique semi-inner product is given by $[x, y] = xy, \, x, y \in \mathbb{R}$. Let $Y = \mathbb{R}^2$ with the max norm and let $f : X \to Y$ be given by

$$f(x) = (x, \sin x), \quad x \in \mathbb{R}.$$ 

Since $f(y)$ is smooth for all $y \in X$, the support functional at $f(y)$ is $\phi_{f(y)}(z) = \frac{y}{|y|}z_1, \, z = (z_1, z_2) \in Y$. Hence for any semi-inner product on $Y$ we have

$$[f(x), f(y)] = \|f(y)\|\phi_{f(y)}(f(x)) = |y|\frac{y}{|y|}x = xy, \quad x, y \in X.$$ 

Therefore, $[f(x), f(y)] = [x, y], \, x, y \in X$, but $f$ is not phase equivalent to any linear isometry.
Example

Let $X = Y = \mathbb{R}^2$ with the max norm, that is, $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$, and let the semi-inner product for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be defined by

$$[x, y] = \begin{cases} x_1y_1 & \text{if } |y_1| \geq |y_2| \\ x_2y_2 & \text{if } |y_1| < |y_2|. \end{cases}$$

Define $f : X \to Y$ by $f((\lambda, \lambda)) = (\lambda, -\lambda)$, $f((\lambda, -\lambda)) = (\lambda, \lambda)$ and $f(x) = x$ for all other directions.

Then $f$ is surjective and $|[f(x), f(y)]| = |[x, y]|$, $x, y \in X$.

However, $f$ is not phase equivalent to any linear isometry from $X$ to $Y$. 
Another variant of Wigner’s theorem in normed spaces

**Theorem**

Let $X$ and $Y$ be normed spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : X \to Y$ a surjective mapping. Suppose that for all semi-inner products on $X$ and $Y$ we have

$$|[f(x), f(y)]| = |[x, y]|, \quad x, y \in X.$$

Then the following holds.

(i) If $\dim X = 1$, then $f$ is phase equivalent to a linear surjective isometry.

(ii) If $\dim X \geq 2$ and $\mathbb{F} = \mathbb{R}$, then $f$ is phase equivalent to a linear surjective isometry.

(iii) If $\dim X \geq 2$ and $\mathbb{F} = \mathbb{C}$, then $f$ is phase equivalent to a linear or an anti-linear surjective isometry.
Key ingredients of the proofs

To prove that for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$ there exist $\omega_1, \omega_2 \in T = \{ t \in \mathbb{F} : |t| = 1 \}$ such that $f(\alpha x + \beta y) = \omega_1 \alpha f(x) + \omega_2 \beta f(y)$.

Birkhoff–James orthogonality: $x \perp y \overset{\text{def}}{\iff} \| x + \lambda y \| \geq \| x \| \ \forall \lambda \in \mathbb{F}$.

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**Theorem (Fundamental theorem of projective geometry)**

*Let $X$ and $Y$ be vector spaces over $\mathbb{F}$ of dimensions at least three. Let $g : \{ \langle x \rangle : x \in X \} \rightarrow \{ \langle y \rangle : y \in Y \}$ be a mapping such that*

(i) The image of $g$ is not contained in a two-dimensional subspace of $Y$.

(ii) $0 \neq c \in \langle a, b \rangle$, $a \neq 0 \neq b$, implies $g(\langle c \rangle) \in \langle g(\langle a \rangle), g(\langle b \rangle) \rangle$.

*Then there exists an injective semilinear mapping $A : X \rightarrow Y$ (that is, $A$ is additive and $A(\lambda x) = h(\lambda)Ax$ for all $x, y \in X$ and $\lambda \in \mathbb{F}$, where $h : \mathbb{F} \rightarrow \mathbb{F}$ is a homomorphism) such that*

$$g(\langle x \rangle) = \langle Ax \rangle, \quad 0 \neq x \in X.$$  

*Moreover, $A$ is unique up to a non-zero scalar factor.*
Another generalization of Wigner’s theorem

Let $X$ and $Y$ be real normed spaces. If $f : X \to Y$ is phase equivalent to a linear isometry, then $f$ satisfies

$$\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\}, \quad x, y \in H.$$  

**Example**

$f : \mathbb{R} \to \mathbb{R}^2$ (with max norm), $f(x) = (x, \sin x)$ satisfies

$$\{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\}, \quad x, y \in \mathbb{R}.$$  

but it is not phase equivalent to any linear isometry.
A generalization of Wigner’s theorem to real smooth normed spaces

Recall that a normed space \( X \) is said to be **strictly convex** whenever the unit sphere \( S_X \) does not contain non-trivial line segments, that is, each point of \( S_X \) is an extreme point of the unit ball \( B_X \).

In a strictly convex normed space the midpoint \( z = \frac{x+y}{2} \) of the segment \( [x, y] \) is characterized by \( \|z - x\| = \|z - y\| = \frac{\|x - y\|}{2} \).

**Proposition**

Let \( X, Y \) be real smooth normed spaces, \( Y \) strictly convex, \( f : X \to Y \) surjective. The following assertions are equivalent:

(i) \( \|[f(x), f(y)]\| = \|[x, y]\|, \ x, y \in X; \)

(ii) \( f \) is phase equivalent to a surjective linear isometry;

(iii) \( \{\|f(x) + f(y)\|, \|f(x) - f(y)\|\} = \{\|x + y\|, \|x - y\|\}, \ x, y \in X. \)
A generalization of Wigner’s theorem to real normed spaces

Theorem

Let $X$ and $Y$ be real normed spaces. Then a surjective mapping $f : X \to Y$ satisfies

$$\{ \| f(x) + f(y) \|, \| f(x) - f(y) \| \} = \{ \| x + y \|, \| x - y \| \}, \quad x, y \in X,$$

if and only if $f$ is phase equivalent to a surjective linear isometry.

Our approach to the problem is motivated by the proof of the Mazur–Ulam theorem on isometries between real normed spaces. We show that the midpoint of the segment $[x, y]$ is mapped to one of the midpoints of the segments $[\pm f(x), \pm f(y)]$. 

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