On the Rank of Pseudo Walk Matrices

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Preliminaries

- Let G be a simple graph on $n = |\mathcal{V}(G)|$ vertices and $\mathbf{A}$ be its 0–1 adjacency matrix.

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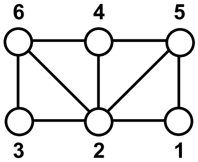

Example: $S = \{(1,1), (1,2), (2,3), (4,6)\}$.

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- Let $S \subseteq \mathcal{V}^2$ and let k be a nonnegative integer. For all $(i,j) \in S$, find the number of walks of length k in G starting from i and ending at j, then sum them up.

- Denote this sum by $N_k(S)$. In other words,

$$N_k(S) = \sum_{(i,j)\in S} [\![\mathbf{A}^k]\!]_{ij}.$$

Walk Matrices

- A walk matrix $\mathbf{W_b}$ is of the form $\begin{pmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \cdots & \mathbf{A}^{n-1}\mathbf{b} \end{pmatrix}$, where $\mathbf{b}$ is a 0–1 vector (usually the all-ones vector $\mathbf{j}$).

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- Question: Given $S \subseteq \mathcal{V}^2$, is there a walk vector $\mathbf{v}$ such that $[\![\mathbf{W_v}^\mathsf{T}\mathbf{W_v}]\!]_{jk} = N_{j+k-2}(S)$ for all j, k?

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- Answer: Yes! (In fact, usually more than one.)

Walk Vectors exist for any S

$(v \quad Av \quad A^2v \quad \cdots \quad A^{n-1}v)$

Theorem

Given any $S \subseteq \mathcal{V}^2$, a walk vector for S is

$$\mathbf{v} = \mathbf{X} \begin{pmatrix} \pm\sqrt{\sum_{(u,v)\in S} [\![\mathbf{X}]\!]_{u1} [\![\mathbf{X}]\!]_{v1}} \\ \pm\sqrt{\sum_{(u,v)\in S} [\![\mathbf{X}]\!]_{u2} [\![\mathbf{X}]\!]_{v2}} \\ \vdots \\ \pm\sqrt{\sum_{(u,v)\in S} [\![\mathbf{X}]\!]_{un} [\![\mathbf{X}]\!]_{vn}} \end{pmatrix}.$$

where $\mathbf{X}$ is an orthogonal matrix that diagonalizes $\mathbf{A}$.

Pseudo Walk Matrices

Definition

A pseudo walk matrix of G associated with $S \subseteq \mathcal{V}^2$ is a matrix

$$\mathbf{W}_{\mathbf{v}} = \begin{pmatrix} \mathbf{v} & \mathbf{A}\mathbf{v} & \mathbf{A}^2\mathbf{v} & \cdots & \mathbf{A}^{n-1}\mathbf{v} \end{pmatrix}$$

where the skew diagonals of $\mathbf{W}_{\mathbf{v}}^{\mathsf{T}}\mathbf{W}_{\mathbf{v}}$ contain the numbers $N_0(S), N_1(S), \ldots, N_{2n-2}(S)$ (from left to right). If the walk vector $\mathbf{v}$ is a 0–1 vector, then $\mathbf{W}_{\mathbf{v}}$ may be simply called a walk matrix.

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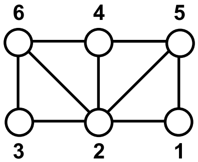
- For some S, the entries of $\mathbf{W_v}$ may not be walk enumerations. Hence the word pseudo (fake).

Example



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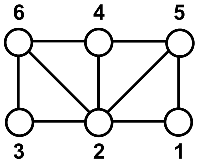

- For $S = \{(1,2)\}$, $\mathbf{v}$ may be chosen to be $\begin{pmatrix} -0.021 - 0.126\mathrm{i} \\ 0.178 - 0.029\mathrm{i} \\ -0.021 - 0.126\mathrm{i} \\ 0.379 - 0.289\mathrm{i} \\ 0.204 + 0.268\mathrm{i} \\ 0.204 + 0.268\mathrm{i} \end{pmatrix}$.

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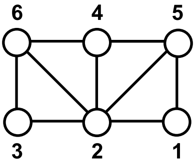

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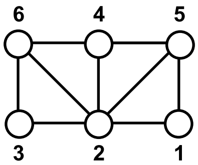
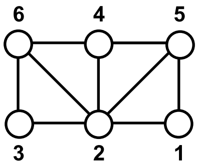
- $\mathbf{W_v} = \begin{pmatrix} \mathbf{v} & \mathbf{Av} & \mathbf{A}^2\mathbf{v} & \mathbf{A}^3\mathbf{v} & \mathbf{A}^4\mathbf{v} & \mathbf{A}^5\mathbf{v} \end{pmatrix}$ with this $\mathbf{v}$.

Example Continued



- $$\mathbf{W_v^T W_v} = \begin{pmatrix} 0 & 1 & 1 & 7 & 16 & 63 \\ 1 & 1 & 7 & 16 & 63 & 183 \\ 1 & 7 & 16 & 63 & 183 & 625 \\ 7 & 16 & 63 & 183 & 625 & 1952 \\ 16 & 63 & 183 & 625 & 1952 & 6401 \\ 63 & 183 & 625 & 1952 & 6401 & 20433 \end{pmatrix}.$$

Example Continued



- $\mathbf{W_v^\mathsf{T} W_v} = \begin{pmatrix} 0 & 1 & 1 & 7 & 16 & 63 \\ 1 & 1 & 7 & 16 & 63 & 183 \\ 1 & 7 & 16 & 63 & 183 & 625 \\ 7 & 16 & 63 & 183 & 625 & 1952 \\ 16 & 63 & 183 & 625 & 1952 & 6401 \\ 63 & 183 & 625 & 1952 & 6401 & 20433 \end{pmatrix}$. It has rank 4.

The Rank of Pseudo Walk Matrices

Theorem

The rank of a pseudo walk matrix $\mathbf{W_v}$ (and of $\mathbf{W_v^\top W_v}$) is the number of eigenvalues of G having an eigenvector not orthogonal to the walk vector $\mathbf{v}$.

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For all walk vectors $\mathbf{v}$, the number of distinct eigenvalues of G is an upper bound for the rank of $\mathbf{W_v}$.

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For all walk vectors $\mathbf{v}$, the number of distinct eigenvalues of G is an upper bound for the rank of $\mathbf{W_v}$.

- This upper bound is reached by the closed pseudo walk matrix.

Closed Pseudo Walk Matrices

- A closed pseudo walk matrix is a pseudo walk matrix $\mathbf{W_v}$ with a walk vector $\mathbf{v}$ associated with $S = \{(1,1),(2,2),\ldots,(n,n)\}$.

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Theorem

The rank of any closed pseudo walk matrix is the number of distinct eigenvalues of G.

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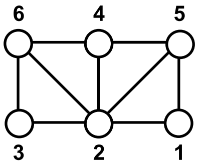

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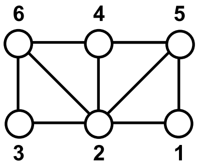

- $S = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.
- $\mathbf{v}$ may be chosen to be the sum of all the orthonormal eigenvectors of G, that is, $\begin{pmatrix}-0.452 & 0.122 & -0.452 & 0.355 & 0.313 & 2.313\end{pmatrix}^{\mathsf{T}}$.

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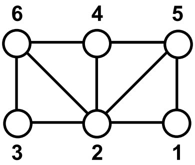

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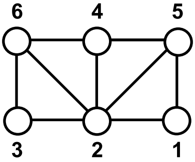
- $\mathbf{W_v}$ and $\mathbf{W_v^{\mathsf{T}} W_v}$ have rank 6.

Example Continued



- $$\mathbf{W}_{\mathbf{v}}^{\mathsf{T}}\mathbf{W}_{\mathbf{v}} = \begin{pmatrix} 6 & 0 & 18 & 24 & 126 & 320 \\ 0 & 18 & 24 & 126 & 320 & 1170 \\ 18 & 24 & 126 & 320 & 1170 & 3528 \\ 24 & 126 & 320 & 1170 & 3528 & 11782 \\ 126 & 320 & 1170 & 3528 & 11782 & 37248 \\ 320 & 1170 & 3528 & 11782 & 37248 & 121298 \end{pmatrix}.$$

Another Restriction on the Rank

- Factorize the characteristic polynomial of G over $\mathbb{Q}$ to obtain $\phi(G,x) = (p_1(x))^{q_1}(p_2(x))^{q_2}\cdots(p_t(x))^{q_t}$.

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Theorem

The rank of any pseudo walk matrix associated with $S \subseteq \mathcal{V}^2$ of a graph G is $d_1 + c_2d_2 + \cdots + c_td_t$, where $c_j \in \{0,1\}$ for all $j \in \{2,\ldots,t\}$. (The c_j's may be different for different pseudo walk matrices.)

Controllable and Recalcitrant Pairs

Corollary

If r is the rank of a pseudo walk matrix associated with some set S of a graph G, then $d_1 \leq r \leq d_1 + \cdots + d_t$.

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Corollary

If $\phi(G, x)$ is irreducible over $\mathbb{Q}$, then $(\mathbf{A}, \mathbf{v})$ is a controllable pair for all walk vectors $\mathbf{v}$.

Graphs with an Irreducible Characteristic Polynomial

Table 1. The number of connected graphs $G(n)$, connected controllable graphs $C(n)$ and connected graphs with an irreducible characteristic polynomial $I(n)$ on n vertices.

n	1	2	3	4	5	6	7	8	9	10
$G(n)$	1	1	2	6	21	112	853	11117	261080	11716571
$C(n)$	1	0	0	0	0	8	85	2275	83034	5512362
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- It is known that $\lim_{n\to\infty} \frac{C(n)}{G(n)} = 1.$
- Conjecture: $\lim_{n\to\infty} \frac{I(n)}{G(n)} = 1.$

Results on Controllable and Recalcitrant Pairs

- Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be indicator vectors of two subsets V_1 and V_2 of $\mathcal{V}(G)$. Moreover, let $\mathbf{v}$ be a walk vector for $V_1 \times V_2$.

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If $(\mathbf{A}, \mathbf{b}_1)$ or $(\mathbf{A}, \mathbf{b}_2)$ is a recalcitrant pair, then the pair $(\mathbf{A}, \mathbf{v})$ is also recalcitrant.

Results on Regular Graphs

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Theorem

If G is a regular graph, then the pair $(\mathbf{A}, \mathbf{v})$ is recalcitrant for any walk vector $\mathbf{v}$ associated with the set $V \times \mathcal{V}(G)$ for all $V \subseteq \mathcal{V}(G)$. Moreover, the pseudo walk matrices of all such walk vectors have rank one.

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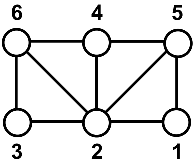
Corollary

If a non-regular graph has its largest eigenvalue equal to an integer, then $(\mathbf{A}, \mathbf{v})$ is not recalcitrant for any pseudo walk vector $\mathbf{v}$.

First Example



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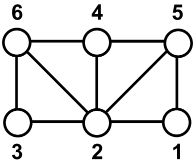

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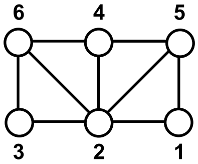

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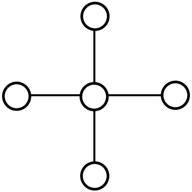
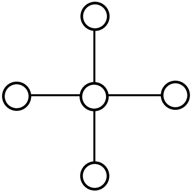
- Since the largest root of $\phi(G,x)$ is a root of $x^4 - 8x^2 - 8x + 1$, every pseudo walk matrix associated with G must have rank 4, 5 or 6.
- In this case, $(\mathbf{A}, \mathbf{v})$ is recalcitrant if $\mathbf{W_v}$ has rank 4; $(\mathbf{A}, \mathbf{v})$ is controllable if $\mathbf{W_v}$ has rank 6.

Second Example



- $\phi(G,x) = x^3(x-2)(x+2)$.

Second Example



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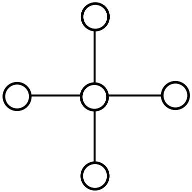

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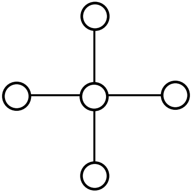

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- Every pseudo walk matrix associated with $K_{1,4}$ has rank 1, 2 or 3.
- However, $K_{1,4}$ is not regular and its largest eigenvalue is an integer, so rank 1 is not possible.
- Thus, for any $\mathbf{v}$, $(\mathbf{A}, \mathbf{v})$ is neither controllable nor recalcitrant.

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- Is there a graph G with a factorizable characteristic polynomial over $\mathbb{Q}$ where, for all walk vectors $\mathbf{v}$, $(\mathbf{A}, \mathbf{v})$ is controllable?

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- Is it true that almost all graphs have an irreducible characteristic polynomial?

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Thank you