Non-commutative polynomial optimization in quantum physics

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Computational aspects of commutative and noncommutative positive polynomials
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Polynomial optimization problems

$$\bar{p} = \min_{x_1, \ldots, x_n} p(x_1, \ldots, x_n)$$

s.t. $q_j(x_1, \ldots, x_n) \geq 0$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

and $p$ and $q_j$ are polynomials of bounded degree $\leq d$. 
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Lasserre and Parrilo introduced a hierarchy of semi-definite programming relaxations whose optima converge to the searched solution.

$$p^{(1)} \leq p^{(2)} \leq \ldots \leq p^{(N)} \rightarrow \bar{p}$$
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Convergence can be proven using Putinar’s result on Positivstellensatz.
Non-commutative polynomial optimization

\[ \bar{p} = \min_{|\psi\rangle, X_1, \ldots, X_n} \langle \psi | P(X_1, \ldots, X_n) | \psi \rangle \]

s.t.  \[ Q_j(X_1, \ldots, X_n) \geq 0 \]

where \( X_1, \ldots, X_n \) are now non-commuting bounded operators, of arbitrary dimension, and \( P \) and \( Q_j \) are Hermitian polynomial operators of bounded degree \( \leq d \).
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Hierarchy of SDP relaxations

We consider the set of monomials of a given degree $\leq k$ on the previous operators, where each monomial is defined by a vector of indices $\alpha$. The degree of the monomial is $|\alpha|=k$.

Example:

$$ Y_{\alpha=(1,3,6,6)} = X_1 X_3^* X_6^2 $$
Hierarchy of SDP relaxations

We consider the set of monomials of a given degree \( \leq k \) on the previous operators, where each monomial is defined by a vector of indices \( \alpha \). The degree of the monomial is \( |\alpha| = k \).

Example:

\[
Y_{\alpha=(1,3,6,6)} = X_1 X_3^* X_6^2
\]

We consider maps from this set to complex numbers:

\[
\Lambda(Y_\alpha) = y_\alpha \in C
\]
Hierarchy of SDP relaxations

For a given sequence, we define the moment matrix $M_k$ of degree $k$ as

$$(M_k)_{\alpha,\beta} = \Lambda \left( X_{\bar{\alpha}} X_{\beta} \right)$$

$|\alpha|, |\beta| \leq k$

$$M_1 = \begin{pmatrix}
1 & y_1 & y_2 & y_{1T} & y_{12} \\
y_{1T} & y_{1T1} & y_{1T2} & y_{1T1} & y_{1T2} \\
y_2 & y_{21} & y_{22} & y_{21T} & y_{22T} \\
y_1 & y_{11} & y_{12} & y_{11T} & y_{112} \\
y_2 & y_{21} & y_{22} & y_{21T} & y_{222}
\end{pmatrix}$$
Hierarchy of SDP relaxations

For a given sequence, we define the moment matrix $M_k$ of degree $k$ as

$$
(M_k)_{\alpha, \beta} = \Lambda(X_{\alpha}X_{\beta})
$$

$$
|\alpha|, |\beta| \leq k
$$

For a polynomial $P(X) = \sum p_\delta X_\delta$ the localizing matrix $L_{P,k}$ of degree $k$ is defined as

$$
(L_{P,k})_{\alpha, \beta} = \sum p_\delta \Lambda(X_{\alpha}X_\delta X_{\beta})
$$

$$
|\alpha|, |\beta| \leq k
$$

$M_k = L_{1,k}$
Hierarchy of SDP relaxations

Every specific choice of operators $X_i$ and state $\psi$ defines a map $\Lambda$:

$$\Lambda(X_\alpha) = \langle \psi | X_\alpha | \psi \rangle$$

For these maps, all localizing matrices defined for positive polynomials are positive.
Hierarchy of SDP relaxations

Every specific choice of operators $X_i$ and state $\psi$ defines a map $\Lambda$:

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Proof:

$$\langle v | L_{P,k} | v \rangle = \sum_{\alpha, \beta} v_\alpha^* (L_{P,k})_{\alpha, \beta} v_\beta = \sum_{\alpha, \beta} v_\alpha^* \sum_\delta p_\delta \Lambda(X_{\bar{\alpha}}X_\delta X_\beta) v_\beta =$$

$$\sum_{\alpha, \beta} v_\alpha^* \sum_\delta p_\delta \langle \psi | X_{\bar{\alpha}}X_\delta X_\beta | \psi \rangle v_\beta = \langle \psi \left| \left( \sum_\alpha v_\alpha^* X_{\bar{\alpha}} \right) \left( \sum_\delta p_\delta X_\delta \right) \left( \sum_\beta v_\beta X_\beta \right) \right| \psi \rangle \geq 0$$
Hierarchy of SDP relaxations

$$\bar{p} = \min \langle \psi | P(X_1, \ldots, X_n) | \psi \rangle$$

s.t. \quad Q_j(X_1, \ldots, X_n) \geq 0$$
Hierarchy of SDP relaxations

\[\bar{p} = \min \langle \psi | P(X_1, \ldots, X_n) | \psi \rangle\]
\[\text{s.t. } Q_j(X_1, \ldots, X_n) \geq 0\]

Relaxation of order \( k \): 

\[p^{(k)} = \min \sum p_\delta y_\delta\]
\[\text{s.t. } M_k \geq 0, L_{Q_j,k-\deg(Q_j)/2} \geq 0\]
Hierarchy of SDP relaxations

\[ p = \min \langle \psi | P(X_1, \ldots, X_n) | \psi \rangle \]
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Clearly:

\[ p^{(1)} \leq p^{(2)} \leq \ldots \leq \bar{p} \]
Convergence of the hierarchy

\[ \bar{p} = \min \langle \psi | P(X_1, \ldots, X_n) | \psi \rangle \]

s.t. \[ Q_j(X_1, \ldots, X_n) \geq 0 \]

Positivity domain: \[ S_Q = \left\{ (X_1, \ldots, X_n) \text{ s.t. } Q_j(X_1, \ldots, X_n) \geq 0 \right\} \]
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Quadratic module:

\[
M_Q = \left\{ P \text{ s.t. } P = \sum_i F_i^* F_i + \sum_{i,j} G_{i,j}^* Q_j G_{i,j} \right\}
\]
Convergence of the hierarchy

\[ \bar{p} = \min \left\langle \psi \left| P(X_1, \ldots, X_n) \right| \psi \right\rangle \]

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\( M_Q \) is Archimedean if \( \exists C \text{ s.t. } C - X_1^* X_1 - \ldots - X_n^* X_n \in M_Q \)
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\bar{p} = \min \langle \psi \left| P(X_1, \ldots, X_n) \right| \psi \rangle
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\]

If \(S_Q\) is bounded, we can make \(M_Q\) Archimedean by choosing a large enough \(C\) and adding to the set of polynomial constraints the condition:

\[
C - X_1^* X_1 - \ldots - X_n^* X_n
\]
Convergence of the hierarchy

If $M_Q$ is Archimedean: $\lim_{k \to \infty} p^{(k)} = \bar{p}$

The proof is constructed from the primal problem of the relaxations.
Convergence of the hierarchy

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The proof is constructed from the primal problem of the relaxations.

Convergence can be established at a finite step whenever the optimal solution $y_k$ for relaxation of order $k$ is such that:

$$\text{rank} \left( M_k \right) = \text{rank} \left( M_{k-\text{max}(d_i)} \right)$$
Convergence of the hierarchy

Consider the problem:

\[ \lambda^{(k)} = \max_{\lambda, B_i, C_{ij}} \lambda \]

s.t. \[ P(\mathbf{X}_1, \ldots, \mathbf{X}_n) - \lambda = \sum_i B_i^* B_i + \sum_{i,j} C_{ij}^* Q_{ij} C_{ij} \]

\[ \max_{i} \text{deg}(B_i) \leq k, \max_{i} \text{deg}(C_{ij}) \leq k - d_i \]
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$$\lambda^{(k)} = \max_{\lambda, B_i, C_{ij}} \lambda$$

s.t.  $$P(X_1, \ldots, X_n) - \lambda = \sum_i B_i^* B_i + \sum_{i,j} C_{i,j}^* Q_{j} C_{i,j}$$

$$\max \text{deg}(B_i) \leq k, \max \text{deg}(C_{i,j}) \leq k - d_i$$

This problem can be cast in a sdp form, and proven to be the dual of step $k$ before.

$$\lambda^{(k)} \leq p^{(k)} \leq \bar{p}$$
Convergence of the hierarchy

\[ P(X_1, \ldots, X_n) - (\bar{p} - \varepsilon) \]

is positive in \( S_Q \) for any \( \varepsilon > 0 \).
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$$P(X_1,\ldots,X_n) - (\bar{p} - \varepsilon)$$ is positive in $S_Q$ for any $\varepsilon > 0$.

Helton and McCullough Positivellensatz

$$P(X_1,\ldots,X_n) - (\bar{p} - \varepsilon) = \sum_i B_i^* B_i + \sum_{i,j} C_{i,j}^* Q_{j} C_{i,j}$$
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Helton and McCullough Positivellensatz

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This defines a feasible point for the previous problem, so one has:

\[ \bar{p} - \varepsilon \leq \lambda^{(k)} \leq p^{(k)} \leq \bar{p} \quad \forall \varepsilon > 0 \]
Relation to classical SDP hierarchies

\[ \bar{p} = \min \langle \psi | P(X_1, \ldots, X_n) | \psi \rangle \]

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\[ [X_i, X_j] = 0 \ \forall i, j \]
Relation to classical SDP hierarchies

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\[p^{(1)} = \bar{p} \quad \text{Easy!}\]

\[
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\text{s.t. } x_i^2 - x_i = 0
\]

NP-hard
Why do we care about this?
Quantum physics

The postulates of quantum theory:
Quantum physics

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1. A complex Hilbert space of dimension $d$ is associated to any physical system. The state of the system is specified by a normalised ray in this space, $|\psi\rangle \in \mathbb{C}^d$ such that $\langle\psi|\psi\rangle = 1$. 
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2. A measurement is defined by a set of orthogonal projectors $M$ acting on the same space, $M = \{M\}_{r=1}^{R}$, such that $\sum_{r=1}^{R} M_r = 1$. 
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3. When implementing the measurement defined by $M$ on a system in state $|\psi\rangle$, result $r$ is obtained with probability $\text{Pr}(r) = \langle \psi | M_r | \psi \rangle$. 
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3. When implementing the measurement defined by $M$ on a system in state $|\psi\rangle$, result $r$ is obtained with probability $\text{Pr}(r) = \langle \psi | M_r | \psi \rangle$.

4. When combining two systems, $A$ and $B$, with corresponding Hilbert spaces $\mathbb{C}^{d_A}$ and $\mathbb{C}^{d_B}$, the Hilbert space of the joint system is the tensor product of the two spaces.
Statistics in quantum experiments

\[ P(r_1, r_2 | p, m_1, m_2) = \text{tr}(\Lambda(\rho_p) M_{r_1}^{m_1} \otimes M_{r_2}^{m_2}) \]
Statistics in quantum experiments

\[ P(r_1, r_2 | p, m_1, m_2) = \text{tr}(\Lambda(\rho_p)M_{r_1}^{m_1} \otimes M_{r_2}^{m_2}) \]

Quantum physics is a natural source of problems involving polynomials of operators.
Characterization of Quantum Correlations

Navascués, Pironio, Acin, PRL 2007, NJP 2009
Physical correlations

The object we deal with is a conditional probability distribution of the outputs given the inputs, which encapsulates the correlations among devices.

$p(a, b|x, y)$
Physical correlations

The object we deal with is a conditional probability distribution of the outputs given the inputs, which encapsulates the correlations among devices.

\[ p(a, b | x, y) = \begin{pmatrix} p(1,1|1,1) & p(1,2|1,1) & \cdots & p(r,r|1,1) \end{pmatrix} \sum_{a,b} p(a,b|1,1) = 1 \]
Physical correlations

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p(1,1|1,2) & p(1,2|1,2) & \cdots & p(r,r|1,2) \\
\end{pmatrix} \]
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p(1,1|1,2) & p(1,2|1,2) & \cdots & p(r,r|1,2) \\
\vdots & \vdots & \ddots & \vdots \\
p(1,1|m,m) & p(1,2|m,m) & \cdots & p(r,r|m,m)
\end{pmatrix}
\]

\[p(a, b | x, y) \in \mathbb{R}^{m^2 r^2}\]

\[p(a, b | x, y) \geq 0\]

\[\sum_{a,b} p(a, b | x, y) = 1\]
Example

\[ p(ab|xy) \]

Alice

\[
\begin{align*}
& x=1,\ldots,m \\
& a=1,\ldots,r
\end{align*}
\]

Bob

\[
\begin{align*}
& y=1,\ldots,m \\
& b=1,\ldots,r
\end{align*}
\]
Example

\[ p(ab \mid xy) = \begin{pmatrix}
  p(+1, +1 \mid 0, 0) & p(+1, -1 \mid 0, 0) & p(-1, +1 \mid 0, 0) & p(-1, -1 \mid 0, 0) \\
  p(+1, +1 \mid 0, 1) & p(+1, -1 \mid 0, 1) & p(-1, +1 \mid 0, 1) & p(-1, -1 \mid 0, 1) \\
  p(+1, +1 \mid 1, 0) & p(+1, -1 \mid 1, 0) & p(-1, +1 \mid 1, 0) & p(-1, -1 \mid 1, 0) \\
  p(+1, +1 \mid 1, 1) & p(+1, -1 \mid 1, 1) & p(-1, +1 \mid 1, 1) & p(-1, -1 \mid 1, 1)
\end{pmatrix} \]
Example

\[ p(+1, +1|0,0) = \frac{1}{2} \]

\[
p(ab|xy) = \begin{pmatrix}
p(+1, +1|0,0) & p(+1, -1|0,0) & p(-1, +1|0,0) & p(-1, -1|0,0) \\
p(+1, +1|0,1) & p(+1, -1|0,1) & p(-1, +1|0,1) & p(-1, -1|0,1) \\
p(+1, +1|1,0) & p(+1, -1|1,0) & p(-1, +1|1,0) & p(-1, -1|1,0) \\
p(+1, +1|1,1) & p(+1, -1|1,1) & p(-1, +1|1,1) & p(-1, -1|1,1)
\end{pmatrix}
\]
Example

\[ p(\text{+1, +1}|0,0) = \frac{1}{2} \]

\[
p(ab|xy) = \begin{pmatrix}
\frac{1}{2} & p(\text{+1, -1}|0,0) & p(\text{-1, +1}|0,0) & p(\text{-1, -1}|0,0) \\
 p(\text{+1, +1}|0,1) & p(\text{+1, -1}|0,1) & p(\text{-1, +1}|0,1) & p(\text{-1, -1}|0,1) \\
 p(\text{+1, +1}|1,0) & p(\text{+1, -1}|1,0) & p(\text{-1, +1}|1,0) & p(\text{-1, -1}|1,0) \\
 p(\text{+1, +1}|1,1) & p(\text{+1, -1}|1,1) & p(\text{-1, +1}|1,1) & p(\text{-1, -1}|1,1)
\end{pmatrix}
\]

Alice

\[ x = 0 \]

\[ a = +1 \]

Bob

\[ y = 0 \]

\[ b = +1 \]
Example

\[ x = 0 \]

Alice

\[ a = -1 \]

Bob

\[ y = 0 \]

\[ b = -1 \]

\[ p(-1, -1|0,0) = \frac{1}{2} \]

\[
p(ab|xy) = \begin{pmatrix}
1/2 & p(+1, -1|0,0) & p(-1, +1|0,0) & 1/2 \\
p(+1, +1|0,1) & p(+1, -1|0,1) & p(-1, +1|0,1) & p(-1, -1|0,1) \\
p(+1, +1|1,0) & p(+1, -1|1,0) & p(-1, +1|1,0) & p(-1, -1|1,0) \\
p(+1, +1|1,1) & p(+1, -1|1,1) & p(-1, +1|1,1) & p(-1, -1|1,1)
\end{pmatrix}
\]
Example

\[ p(+1, -1|0,0) = p(-1, +1|0,0) = 0 \]

\[
p(ab|xy) = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
p(+1, +1|0,1) & p(+1, -1|0,1) & p(-1, +1|0,1) & p(-1, -1|0,1) \\
p(+1, +1|1,0) & p(+1, -1|1,0) & p(-1, +1|1,0) & p(-1, -1|1,0) \\
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Example

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\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & \frac{1}{2}
\end{pmatrix} \]

Alice: \( x=0 \) and \( a=+1,-1 \)
Bob: \( y=1 \) and \( b=+1,-1 \)
Example

\[ p(ab|xy) = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
\end{pmatrix}
\]

Alice $x=1$, $a=+1,-1$

Bob $y=0$, $b=+1,-1$
Example

\[ p(ab|xy) = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix} \]
Physical correlations

Physical principles translate into limits on correlations.
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**No-signalling correlations**: correlations compatible with the no-signalling principle, i.e. the impossibility of instantaneous communication.

\[
\sum_{a_{k+1}, \ldots, a_N} p(a_1, \ldots, a_N | x_1, \ldots, x_N) = p(a_1, \ldots, a_k | x_1, \ldots, x_k)
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\[ p_A(+1|0) = p(+1, +1|00) + p(+1, -1|00) = \frac{1}{2} \]
**Physical correlations**

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\[
p_A(+1|0) = p(+1, +1|00) + p(+1, -1|00) = \frac{1}{2} = p(+1, +1|01) + p(+1, -1|01)
\]
Physical correlations

**Classical correlations**: correlations established by classical means.

\[
p(a_1, \ldots, a_N | x_1, \ldots, x_N) = \sum_{\lambda} p(\lambda) D(a_1 | x_1, \lambda) \cdots D(a_N | x_N, \lambda)
\]

These are the standard “EPR” correlations. Independently of fundamental issues, these are the correlations achievable by classical resources. Bell inequalities define the limits on these correlations.
Physical correlations

Quantum correlations: correlations established by quantum means.

\[ p(a_1,\ldots,a_N \mid x_1,\ldots,x_N) = \langle \Psi \mid M_{a_1}^{x_1} \otimes \cdots \otimes M_{a_N}^{x_N} \mid \Psi \rangle \]

\[ \sum_{a_i} M_{a_i}^{x_i} = 1 \quad M_{a_i'}^{x_i} M_{a_i}^{x_i} = \delta_{a_i a_i'} M_{a_i}^{x_i} \]

Everything is expressed in terms of operators (the quantum state and the measurement projectors) acting on a Hilbert space.
Physical correlations

\[ C \subset Q \subset NS \]

\[ p(a, b | x, y) \]
There exist correlations that cannot be explained by a classical model in which (deterministic) classical instructions specify the outcomes of the devices. These quantum correlations are known as **non-local** and they are detected by the violation of a Bell inequality.
There exist correlations that are compatible with the no-signalling principle but cannot be obtained by performing local measurements on a quantum (entangled) state.

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Physical correlations

\[ C \subset Q \subset NS \]

Bell

Tsirelson

Popescu-Rohrlich
Example: CHSH scenario

$$CHSH = A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2$$
Example: CHSH scenario

\[ CHSH = A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 \]

\[
\begin{pmatrix}
0.5 & 0 & 0 & 0.5 \\
0.5 & 0 & 0 & 0.5 \\
0.5 & 0 & 0 & 0.5 \\
0 & 0.5 & 0.5 & 0
\end{pmatrix}
\]

\[ CHSH \leq 4 \]
\[ CHSH \leq 2\sqrt{2} \]
\[ CHSH \leq 2 \]
Characterizing quantum correlations

Given \( p(a,b|x,y) \), does it have a quantum realization?

\[
p(a,b|x,y) = \langle \Psi | M_a^x \otimes M_b^y | \Psi \rangle \quad \text{and} \quad \sum_a M_a^x = 1
\]

\[
M_a^x M_a^{x'} = \delta_{a,a'} M_a^x
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Characterizing quantum correlations

Given $p(a,b|x,y)$, does it have a quantum realization?

$$p(a,b|x,y) = \langle \Psi | M_a^x \otimes M_b^y | \Psi \rangle$$

$$\sum_a M_a^x = 1$$

$$M_a^x M_{a'}^x = \delta_{a'a} M_a^x$$

Example:

$$p(a,b|0,0) = p(a,b|0,1) = p(a,b|1,0) = \frac{1}{8} \left( 2 + \sqrt{3}, 2 - \sqrt{3}, 2 - \sqrt{3}, 2 + \sqrt{3} \right)$$

$$p(a,b|1,1) = (0.245, 0.255, 0.255, 0.245)$$

Previous work by Tsirelson
NPA hierarchy

Given a probability distribution $p(a,b|x,y)$, we have defined a hierarchy consisting of a series of tests based on semi-definite programming techniques allowing the detection of supra-quantum correlations.

\[ \gamma_1 \geq 0 \quad \Rightarrow \quad \gamma_2 \geq 0 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \gamma_\infty \geq 0 \]

The hierarchy is asymptotically convergent.
Every step in the hierarchy defines a convex set that is included in the previous step. Convergence is provably attained asymptotically.
Characterizing quantum correlations

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\[ p(a, b|1,1) = (0.245, 0.255, 0.255, 0.245) \]

Solution: it is not quantum, that is, there exists no quantum state of two particles and local measurements acting on them that produce these correlations.

The experimental observation of these correlations would imply the failure of quantum physics, as Bell violations did for classical physics.
Going beyond NPA
Ground-state energies

A standard problem in physics is to find the ground state energy of a systems of $N$ particles whose interactions are described by a Hamiltonian operator $H$.

$$\min_{|\psi\rangle} \langle \psi | H | \psi \rangle \quad \text{subject to some constraints}$$
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\min_{|\psi\rangle} \langle \psi | H | \psi \rangle \quad \text{subject to some constraints}
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Variational approach: often, one can guess good candidates to solve this problem. The minimization is performed over a subset of states \( \Rightarrow \) upper bound.
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\min_{\psi} \langle \psi | H | \psi \rangle \quad \text{subject to some constraints}
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Variational approach: often, one can guess good candidates to solve this problem. The minimization is performed over a subset of states \(\Rightarrow\) upper bound.

If the problem can be cast as a non-commutative polynomial optimization, the previous hierarchy provides lower bounds. It complements the standard approach!
Classical spin systems

Classical spin problems:

\[
\min_{\vec{\sigma}} H(\vec{\sigma}) = \sum_{i,j} J_{i,j} \sigma_i \sigma_j + \sum_i h_i \sigma_i
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such that:

\[\sigma_i^2 - 1 = 0 \text{ for all } i\]

Commutative polynomial optimization:

\[E_1 \leq E_2 \leq \cdots \leq E_\infty \rightarrow E_g \leq E_a\]
Quantum spin problems:

$$\min_{\hat{\sigma}} H(\hat{\sigma}) = \sum_{i,j} J_{i,j} \sigma_i^x \sigma_j^x + \sum_i h_i \sigma_i^z$$

such that:

$$\sigma_i^\alpha - 1 = 0$$

$$[\sigma_j^\alpha, \sigma_k^\beta] = 2i \delta_{jk} \epsilon_{\alpha \beta \gamma} \sigma_k^\gamma$$

Non-commutative polynomial optimization:

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Non-commutative polynomial optimization:

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E_1 \leq E_2 \leq \cdots \leq E_\infty \rightarrow E_g \leq E_a
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See Jie Wang’s talk.
Main question: understand the causes that could be behind the observed correlations among a set of random variables.
Causal networks

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Given two correlated variables, either direct causation is possible.
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Given two correlated variables, either direct causation is possible.

But even more intricate causation patterns could explain the correlations.
Causal networks

Representation of causality patterns through directed acyclic graphs. Observed variables are represented by circles, hidden variables by squares and causes by directed edges.
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Bell setups can be understood in this language. Fritz, NJP’12; Wood & Spekkens, NJP ‘15
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Quantum causality

Bell’s theorem: nonlocal correlations can be explained by a quantum causal model, but not by the classical counterpart.

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Bell’s theorem: nonlocal correlations can be explained by a quantum causal model, but not by the classical counterpart.

\[ p(a, b|x, y) \]
How is causality affected by quantum information?

See Alejandro Pozas-Kerstjens’ talk.
Quantum information technologies

Quantum Computer

Quantum Simulator

Quantum Cryptography

QRNG
Quantum certification

Is this a Quantum Computer?

Does this properly simulate a quantum system?

Is this cryptographically secure?

Is this quantum random?