On a weighted inequality for fractional integrals

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Portorož, June 23, 2021
We establish necessary and sufficient condition on a non-negative locally integrable function \( \nu \) guaranteeing the (trace) inequality

\[
\| I_\alpha f \|_{L^p_v(\mathbb{R}^n)} \leq C \| f \|_{L^{p,1}(\mathbb{R}^n)}
\]

for the Riesz potential \( I_\alpha \), where \( L^{p,1}(\mathbb{R}^n) \) is the Lorentz space. The same problem is studied for potentials defined on spaces of homogeneous type.
Introduction

Trace inequalities for Riesz potentials $I_\alpha$ deals with non-negative measures $\nu$ such that

$$\left( \int_{\mathbb{R}^n} |I_\alpha f(x)|^q d\nu \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.$$ (0.1)

D. Adams [1] proved that necessary and sufficient condition on $\nu$ guaranteeing (0.1) for $1 < p < q < \infty$ and $0 < \alpha < n/p$ is that measure $\nu$ satisfies the condition: there is a positive constant $C$ such that for all balls $B \subset \mathbb{R}^n$,

$$\nu(B) \leq C |B|^{(\alpha/n - \frac{1}{p})q}.$$
Riesz potential operator

\[ I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \quad x \in \mathbb{R}^n, \]

plays an important role in PDEs. It is worth mentioning its role in the theory of Sobolev’s embeddings (see, e.g., V. G. Maz’ya [12]).
The appropriate fractional maximal operator is given by the formula:

\[ M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)| \, dy, \quad 0 \leq \alpha < n, \quad x \in \mathbb{R}^n. \]

\( M_0 f = Mf \) is the Hardy–Littlewood maximal function having great importance in Harmonic Analysis for example, in the theory of Singular integrals.
Let $\nu$ be a non-negative locally integrable function on $\mathbb{R}^n$. We are interested in the inequality (0.1) for $d\nu = vdx$, i.e.

$$\| I_{\alpha} f \|_{L_v^q(\mathbb{R}^n)} \leq C \| f \|_{L_p(\mathbb{R}^n)}. \quad (0.2)$$
In this case by the result of D. Adams [1] the condition

$$ [v]_{p,q,\alpha} := \sup_B \left( v(B) \right)^{1/q} |B|^{\frac{\alpha}{n} - \frac{1}{p}} < \infty, $$  \hspace{1cm} (0.3)

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, is simultaneously necessary and sufficient whenever $1 < p < q < \infty$ and $0 < \alpha < n/p$. In the case $p = q$ the implication $(0.2) \implies (0.3)$ can be checked easily by considering the test functions $\chi_B$; however the fact that $(0.3) \implies (0.2)$ is not true (see appropriate counterexamples in D. R. Adams [2], R. Kerman and E. Sawyer [14] for a measure $\nu$, and P. G. Lemarié-Rieusset [8] for non-negative function $v$).
Our aim is to find a Lorentz space $L^{p,s}$, which is narrower than the class $L^p(\mathbb{R}^n)$ (i.e., $s < p$) and for which the inequality

$$\|I_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq C\|f\|_{L^{p,s}(\mathbb{R}^n)}$$

holds if and only if (0.3) is satisfied for $p = q$. In particular we show that (0.4) is equivalent to the condition (0.3) for $s = 1$. The question for $1 < s < p$ remains open.
It should be mentioned that there are known various different criteria for (0.2) with \( p = q \) (see D. R. Adams [2], V. G. Maz’ya [10], V. G. Maz’ya [11], R. Kerman and E. Sawyer [14], V. G. Maz’ya and I. Verbitsky [13]). For the solution of the two-weight problem for Riesz potential operators \( I_\alpha \) we refer to M. Gabidzashvili and V. Kokilashvili [6], E. Sawyer [15] (see also the monograph V. Kokilashvili and M. Krbec [7]). Inequality (0.2) for \( p = q \) implies the estimate:

\[
\|f\|_{L^q_v(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C^\infty_0, \tag{0.5}
\]

which follows from the estimate

\[
|f(x)| \leq C l_1(|\nabla f|)(x).
\]
The following Fefferman-Phong C. Fefferman [4] type theorem holds:

**Theorem (A)**

Let $1 < p < \infty$ and let $0 < \alpha < n/p$. Then the following inequality holds:

$$
\| I_\alpha f \|_{L^p_v} \leq C[v]^{*,r,\alpha}_{p} \| f \|_{L^p}
$$

for some $p < r$, where

$$
[v]^{*,r,\alpha}_{p} := \sup_B |B|^{\frac{\alpha}{n}-\frac{1}{r}} \left( \int_B v^{r/p}(x)dx \right)^{1/r} < \infty. \quad (0.6)
$$

**Remark 1:** It is easy to see that by Hölder’s inequality we have that condition (0.6) is stronger than (0.3) for $p = q$, in particular, $[v]_{p,\alpha} \leq [v]^{*,r,\alpha}_{p}$ for $r > p$, where $[v]_{p,\alpha} = [v]_{p,p,\alpha}$. 
Let $f$ be a measurable function on $\mathbb{R}^n$ and let $1 \leq p < \infty$, $1 \leq s \leq \infty$. We say that $f$ belongs to the Lorentz space $L^{p,s}$ if

$$
\|f\|_{L^{p,s}} = \begin{cases} 
\left( s \int_0^\infty \left( |\{x \in \mathbb{R}^n : |f(x)| > \tau\}| \right)^{s/p} \tau^{s-1} d\tau \right)^{1/s}, & \text{if } 1 \leq s < \infty, \\
\sup_{s>0} s \left( |\{x \in \mathbb{R}^n : |f(x)| > s\}| \right)^{1/p}, & \text{if } s = \infty
\end{cases}
$$

is finite.

If $p = s$, then $L^{p,s}$ coincides with the weighted Lebesgue space $L^p$.

It is worth mentioning, that if $1 \leq p < \infty$, $s_2 \leq s_1$, then $L^{p,s_2} \hookrightarrow L^{p,s_1}$ with the embedding constant $C_{p,s_1,s_2}$ depending only on $p$, $s_1$ and $s_2$;
Main Result

Theorem (1)

Let $1 < p < \infty$ and let $0 < \alpha < n/p$. Then the following statements are equivalent:

(i) there is a positive constant $C$ such that for all $f \in L^{p,1}(\mathbb{R}^n)$,

$$\|l_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq C\|f\|_{L^{p,1}(\mathbb{R}^n)} \quad (0.7)$$

(ii) there is a positive constant $c$ such that for all $f \in L^{p,1}(\mathbb{R}^n)$,

$$\|M_\alpha f\|_{L^p_v(\mathbb{R}^n)} \leq c\|f\|_{L^{p,1}(\mathbb{R}^n)} \quad (0.8)$$

(iii) $[v]_{p,\alpha} = \sup_B (v(B))^{1/p} |B|^\frac{\alpha}{n} - \frac{1}{p} < \infty$.

Moreover, if $C$ and $c$ are best constant in (0.7) and (0.8) respectively, then

$$C \approx c \approx [v]_{p,\alpha}.$$
Let \((X, d, \mu)\) be a quasi-metric measure space with a quasi-metric \(d\) and measure \(\mu\). A quasi-metric \(d\) is a function \(d: X \times X \to [0, \infty)\) which satisfies the following conditions:

(i) \(d(x, y) = 0\) if and only if \(x = y\);
(ii) for all \(x, y \in X\), \(d(x, y) = d(y, x)\);
(iii) there is a positive constant \(\kappa\) such that

\[
d(x, y) \leq \kappa (d(x, z) + d(z, y))
\]

for all \(x, y, z \in X\).
In what follows we will assume that the balls

\[ B(x, r) := \{ y \in X; d(x, y) < r \} \]

are measurable with positive \( \mu \) measure for all \( x \in X \) and \( r > 0 \).

If \( \mu \) satisfies the doubling condition:

\[ \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)), \quad (0.9) \]

with a positive constant \( C_\mu \) independent of \( x \) and \( r \), then we say that

\( (X, d, \mu) \)

is a space of homogeneous type (\( SHT \)). We will assume that

\( (X, d, \mu) \)

is an \( SHT \).
The case of Spaces of Homogeneous Type

For example, rectifiable curves in $\mathbb{C}$ with Euclidean distance and arc-length measure satisfying Carleson (regularity) condition, nilpotent Lie groups with Haar measure, domains in $\mathbb{R}^n$ with so-called $\mathcal{A}$ condition are examples of an $SHT$. For the definition, examples and some properties of an $SHT$ see, e.g., the paper R. A. Macías and C. Segovia [9] and the monographs J. O. Strömberg and A. Torchinsky [16], R. R. Coifman and G. Weiss [3].

For a given quasi-metric measure space $(X, d, \mu)$ and $q$ satisfying $1 \leq q \leq \infty$, as usual, we will denote by $L^q = L^q(X, \mu)$ the Lebesgue space equipped with the standard norm. Let $L^{p,s}(X, \mu)$ be the Lorentz space defined on an $SHT$ $(X, d, \mu)$. 
Let us denote by \( K_\alpha f \) Riesz potential of a \( \mu \)-measurable function \( f \) given by the formula:

\[
K_\alpha f(x) = \int_X \mu(B_{xy})^{\alpha - 1} f(y) \, d\mu(y), \quad x \in X,
\]

where \( 0 < \alpha < 1, \ B_{xy} := B(x, d(x, y)) \).

The appropriate fractional maximal function has the form

\[
M_\alpha f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| \, d\mu(y), \quad x \in X.
\]
The case of Spaces of Homogeneous Type

The following trace inequality for an $SHT$ was proved by Gabidzashvili (see [5]).

**Theorem (B)**

Let $1 < p < q < \infty$ and let $0 < \alpha < 1/p$. Suppose that $(X, d, \mu)$ is an $SHT$ and $\nu$ is another measure on $X$. Then the inequality

$$\|K_\alpha f\|_{L^q(X, \nu)} \leq C\|f\|_{L^p(X, \mu)}$$

holds if and only if

$$\sup_{B} \left( \frac{\nu B}{\mu B} \right)^{1/q} \mu(B)^{\alpha-1/p} < \infty.$$

Analyzing the proof of Theorem (1) we can formulate the same result for an $SHT$. In particular, the following Theorem holds:
The case of Spaces of Homogeneous Type

Theorem (2)

Let $1 < p < \infty$ and let $0 < \alpha < 1/p$. Suppose that $(X, d, \mu)$ be an SHT. Assume that $v$ is non-negative $\mu$ locally integrable function on $X$. Then the following statements are equivalent:

(i) there is a positive constant $C$ such that for all $f \in L^{p,1}(X, \mu)$,

$$
\|K_\alpha f\|_{L^p_v(X,\mu)} \leq C\|f\|_{L^{p,1}(X,\mu)}; \quad (0.10)
$$

(ii) there is a positive constant $c$ such that for all $f \in L^{p,1}(X, \mu)$,

$$
\|M_\alpha f\|_{L^p_v(X,\mu)} \leq c\|f\|_{L^{p,1}(X,\mu)}; \quad (0.11)
$$

(iii) $[v]_{p,\alpha,X,\mu} = \sup_B \left( \int_B v(x) d\mu(x) \right)^{1/p} \mu(B)^{\alpha-\frac{1}{p}} < \infty$.

Moreover, if $C$ and $c$ are best constants in (0.10) and (0.11) respectively, then $C \approx c \approx [v]_{p,\alpha,X,\mu}$. 
References


Acknowledgement

The work was supported by the Shota Rustaveli National Science Foundation of Georgia (Project No. DI-18-118).
Thank you for your attention!