On maps preserving products

Louisa Catalano

Union College

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Maps preserving products

Let A be an algebra and fix $a, b \in A$ and let $f : A \to A$ be an additive map. Assume that

$$f(x)f(y) = b \text{ whenever } xy = a$$

for all $x, y \in A$.

Goal: to find the form of f.

Motivation – maps preserving the zero product

We say that an additive map $f : A \to A$ preserves the zero product if

$$f(x)f(y) = 0 \text{ whenever } xy = 0.$$

This has been studied when A is a

- simple algebra (Wong, 1980)
- von Neumann algebra (Cui, Hou, 2002)
- prime ring (Chebotar, Ke, Lee, 2004; Brešar, 2007)
- group algebra (Alaminos, Extremera, Villena, 2015)

In each case, the authors find that $f(x) = \lambda\varphi(x)$ for all $x \in A$, where $\lambda \in A$ is central and φ is a homomorphism.

Motivation – maps preserving more general products

Let $f : A \to A$ be an additive map such that

$$f(x)f(x^{-1}) = f(y)f(y^{-1})$$

for all invertible $x, y \in A$.

In 2005, Chebotar, Ke, Lee, and Shiao found that when f is bijective and A is a division ring, f must have the form $f(x) = f(1)\varphi(x)$, where $f(1)$ is central and φ is an automorphism or antiautomorphism.

In 2006, Lin and Wong generalized this in the case where A is the ring of $n \times n$ matrices over a division ring.

Maps preserving products equal to invertible elements

The following question was posed by Chebotar, Ke, Lee, and Shiao:

Question

Let A be a division algebra and let $a \in A$ be a nonzero element. Suppose that $f : A \to A$ is a bijective additive map such that

$$f(x)f(y) = f(u)f(v) \text{ whenever } xy = uv = a.$$

Is it possible to describe f?

Maps preserving products equal to invertible elements

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Is it possible to describe f?

Yes, it is!

Maps preserving products equal to invertible elements

Theorem (L. C., 2018)

Let A be a division ring with characteristic different from 2 and with center Z. Let $a, b \in A$ be nonzero fixed elements, and let $f : A \to A$ be a bijective additive map satisfying the identity

$$f(x)f(y) = b \text{ whenever } xy = a$$

for every $x, y \in A$. Then $f(x) = f(1)\varphi(x)$ for all $x \in A$, where $\varphi : A \to A$ is either an automorphism or an antiautomorphism. Moreover, we have

1. if φ is an automorphism then $f(1) \in Z$, and
2. if φ is an antiautomorphism, then $f(1) = f(a)^{-1}b$, and $f(a) \in Z$.

Example

Let $A = \mathbb{H} = \langle \mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ be the algebra of quaternions. We will consider the 2×2 complex matrix representations of the elements of $\mathbb{H}$; any element $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ can be written as

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}.$$

Consider the map $f : \mathbb{H} \to \mathbb{H}$ given by

$$f(x) = \mathbf{i}x^{\mathsf{T}},$$

where x^{T} represents the transpose of matrix x. We have that

$$f(x)f(y) = -\mathbf{i} \text{ whenever } xy = \mathbf{i};$$

however, $f(\mathbf{1}) = \mathbf{i}$ is not a central element, but $f(\mathbf{i}) = -\mathbf{1}$ is.

Maps preserving products equal to invertible elements

We were able to generalize our previous result to the case of $n \times n$ matrices over a division ring.

Theorem (L. C., S. Hsu, R. Kapalko, 2019)

Let D be a division ring with characteristic 0. Let $A = M_n(D)$ be the ring of $n \times n$ matrices with $n \geq 2$, and let Z be the center of A. With $a, b \in A$ invertible fixed elements, let $f : A \to A$ be a bijective additive map satisfying the identity

$$f(x)f(y) = b \text{ whenever } xy = a$$

for every $x, y \in A$. Then $f(x) = f(1)\varphi(x)$ for all $x \in A$, where $\varphi : A \to A$ is either an automorphism or an antiautomorphism. Moreover, we have

1. if φ is an automorphism, then $f(1) \in Z$, and
2. if φ is an antiautomorphism, then $f(1) = f(a)^{-1}b$ and $f(a) \in Z$.

What about elements "between" 0 and invertible?

- Zero product $\rightarrow$ central element $\times$ homomorphism
- Invertible product $\rightarrow$ fixed (not necessarily central) element $\times$ homomorphism or antihomomorphism

Question

What about maps preserving products equal to nonzero, noninvertible elements?

We will consider this question on $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices over the complex numbers.

By e_{ij}, we denote the matrix that has a 1 in the ij-entry and zeros elsewhere.

Maps preserving products equal to e_{11}

For the first "test" case, we considered the identity

$$f(x)f(y) = e_{11} \text{ whenever } xy = e_{11}.$$

Theorem (L. C., 2021)

Let a, b be fixed rank-one idempotents of $M_n(\mathbb{C})$ and let $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map satisfying the property that

$$f(x)f(y) = b \text{ whenever } xy = a.$$

Then $f(x) = \lambda u^{-1}xu$ for every $x \in M_n(\mathbb{C})$, where $\lambda = \pm 1$ and u is an invertible element of $M_n(\mathbb{C})$.

Maps preserving products equal to e_{12}

Next, we studied the property

$$f(x)f(y) = e_{12} \text{ whenever } xy = e_{12}.$$

Theorem (L. C., M. Chang-Lee, 2019)

Let $n \geq 4$ and fix $a, b \in M_n(\mathbb{C})$ rank-one nilpotent matrices. Let $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear, bijective map such that

$$f(x)f(y) = b \text{ whenever } xy = a.$$

Then $f(x) = \lambda u^{-1}xu$ for some invertible $u \in M_n(\mathbb{C})$ and nonzero $\lambda \in \mathbb{C}$.

Comparing these results

Although the previous two results seem similar, the techniques used to prove them are quite different.

For the rank-one idempotent case, we relied heavily on the matrix units e_{ij} and previous works on zero product preserving maps.

For the rank-one nilpotent case, we used zero-2 pairs:

Zero-2 Pairs

Let $\mathcal{X}_2$ be the set of all matrices with all entries zero except at most two entries from the same row. Let $\mathcal{Y}_2$ be the set of all matrices with all entries zero except at most two entries from the same column. By a **zero-2 pair**, we denote a pair $(x, y) \in \mathcal{X}_2 \times \mathcal{Y}_2$ such that $xy = 0$.

How to generalize?

Question

Can we use the Jordan normal form to generalize these results?

Goal: get results for

- diagonal matrices, and
- nilpotent Jordan blocks

and "stitch" these results together.

Maps preserving products equal to diagonalizable matrices

Our next result was considering products equal to diagonal matrices. This emphasizes the differences found in the cases of full rank and rank-$(n-1)$.

Theorem (L. C., H. Julius, 2021)

Let $d_1, d_2 \in M_n(\mathbb{C})$ be diagonalizable matrices with the same eigenvalues, counting multiplicities. Let $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map satisfying the property

$$f(x)f(y) = d_2 \text{ whenever } xy = d_1.$$

If $\text{rank}(d_1) = \text{rank}(d_2) < n$, then $f(x) = \lambda u^{-1} x u$ for every $x \in M_n(\mathbb{C})$, where $\lambda \in \mathbb{C}$ is nonzero and $u \in M_n(\mathbb{C})$ is invertible.

If $\text{rank}(d_1) = \text{rank}(d_2) = n$, then either $f(x) = \lambda u^{-1} x u$ or $f(x) = u x^T v$ for every $x \in M_n(\mathbb{C})$, where $\lambda \in \mathbb{C}$ is nonzero and $u, v \in M_n(\mathbb{C})$ are invertible.

However, we had trouble "stitching" this result with that of a nilpotent Jordan block.

Maps preserving products more generally

Transferring our focus, we were able to recover more general results, the first part of which relied on zero-2 pairs.

Theorem (L. C., H. Julius, 2021)

Let $a, b \in M_n(\mathbb{C})$ be a fixed matrices, and let $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map such that

$$f(x)f(y) = b \text{ whenever } xy = a.$$

If we have either

1. $\operatorname{rank}(a) \leq n-2$, or
2. $\operatorname{rank}(a) = \operatorname{rank}(b) = n-1$,

then there exists a nonzero $\lambda \in \mathbb{C}$ and invertible $u \in M_n(\mathbb{C})$ such that $f(x) = \lambda u^{-1}xu$ for all $x \in M_n(\mathbb{C})$.

Future Research

In the previous theorems we have descriptions of the map f preserving products in most cases. What about the remaining ones?

- Do the rank of a and b have to be the same?
- If not, what techniques do we need to get a classification of f for the remaining cases?

THANK YOU!