Hydrodynamic stability for the dynamic slip boundary condition

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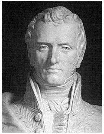
Navier-Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u} = -\nabla p + \mathbf{f} \text{ in } (0, T) \times \Omega$$
$$\text{div } \mathbf{u} = 0$$
$$\mathbf{u}(0, x) = \mathbf{u}_0(x)$$
$$\alpha(\mathbf{u} + \eta) + \beta \partial_t \mathbf{u} = -\nu \nabla \mathbf{u} \cdot \mathbf{n} \text{ on } (0, T) \times \partial\Omega$$



C. L. Navier

- $\Omega \subset \mathbb{R}^3$ (un)bounded
- Dynamical bdd condition ($\alpha, \beta > 0$)
- Goal: long-time behaviour (linear (in)stability)
- There exists weak solution satisfying en. inequality

Sir G. Stokes

Linearization

- nice stationary solution $\mathbf{u}^* = \mathbf{u}^*(x)$ - does **not** depend on β
- arbitrary perturbation $\mathbf{v}$ with the same $\mathbf{f}, \eta$ - depends on β
- consider $\mathbf{u} := \mathbf{v} - \mathbf{u}^*$:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}^* + \mathbf{u}^* \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u} = -\nabla p$$
$$\alpha \mathbf{u} + \beta \partial_t \mathbf{u} = -\nu \nabla \mathbf{u} \cdot \mathbf{n}$$

- linearization = we omit (single) nonlinear term:

$$\partial_t \mathbf{u} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{u}^* + \mathbf{u}^* \cdot \nabla \mathbf{u} - \nu \triangle \mathbf{u}}_{"-\mathcal{L}"} = -\nabla p \text{ in } (0,T) \times \Omega$$

Theorem (Sattinger 1970)

Assume that any $\lambda \in \sigma(\mathcal{L})$ has negative real part. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $||u(0)||_2 < \delta$ implies $||u(t)||_2 < \varepsilon, \forall t > 0$. Moreover $||u(t)||_2 \to 0$ exponentially.

About the proof

- Modification for Dynamical bdd
- $\mathbf{u}^*$ and its dependence on β
- Prodi (1962), Yudovich (1965) - only strong solutions
- Boundedness of Ω
- $\mathcal{L}$ is Hilbert-Schmidt and this gives basis
- "Galerkin" $+$ energy inequality

Stationary solutions

- We need specific $\mathbf{u}^*$ to compute something
- $\Omega =$ two infinite parallel plates ($z \in [0, h]$)
  - Plates are moving and the flow is unidirectional (Couette), $\mathbf{u}^* = (U(z), 0, 0)$
  - Plates are still and the pressure gradient is applied in a direction parallel to the plates (Poiseuille)
- $\Omega =$ two infinite concentric cylinders ($0 \leq R_1 \leq R_2$)
  - Both cylinders are rotating (Taylor-Couette)
  - Pressure as above (Poiseuille)
- Open questions even for simple Dirichlet bdd

Sir G. I. Taylor

Spectrum for the Couette flow I

- Normal modes are complete in the space of 2D-perturbations

$$\mathbf{v}(t, \mathbf{x}) = e^{\sigma t} e^{iAx} \omega(z), \sigma \in \mathbb{C}, A > 0$$

- We want some condition which guarantee that $\Re\sigma < 0$
- We obtain (for example)

$$\Re\sigma < \frac{\max |U'|}{2A} - \nu \frac{4\pi^2 + A^2}{A}$$

- For $\max |U'|/72.26 < \nu$ we get the stability
- Optimal estimates for $\int |\varphi^{(k)}|^2$ - best constant
- Squire theorem: stable in 2D $\implies$ stable in 3D

Spectrum for the Couette flow II

- Romanov (1973) gives stability also for small values of ν
- Altogether unconditional stability, i. e. stable for any ν
- Estimate $\max |U'|/72.26 < \nu$ also for cylinders and other flows
- It also works for dynamical bdd condition
- $\beta-$indepdendet estimate (stationary solutions ignores β)
- But the perturbation $\mathbf{v}$ depends on β, so it gives something
- We would like to have $\mathbf{u}^*$ also time dependent...

Summary and Questions for the future

- We have linearization principle for nice bounded domains
- Similar result for unbounded ones?
- We are able to find conditions to guarantee stability of $\mathbf{u}^*$
- Instability?
- Stationary solution $\rightarrow$ periodical solution?
- Short-time behaviour (pseudospectra)?