Noncommutative ergodic theory of higher rank lattices

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Higher rank lattices

Let G be any **real connected semisimple Lie group** with finite center, no compact factor and real rank ≥ 2.

Let $\Gamma < G$ be any **irreducible lattice**, meaning that $\Gamma < G$ is a discrete subgroup with finite covolume such that $\Gamma \cdot N < G$ is dense for every noncentral closed normal subgroup $N \lhd G$.

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Examples (Minkowski, Borel–Harish-Chandra)

- If $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$, take $\Gamma = \mathrm{SL}_n(\mathbb{Z})$
- If $G = \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ for $n \geq 2$, take $\Gamma = \mathrm{SL}_n(\mathbb{Z}[\sqrt{d}])$ where $d \in \mathbb{N}$ is square free.

In this talk, we simply say that $\Gamma < G$ is a **higher rank lattice**.

Motivation

Margulis' Normal Subgroup Theorem (1978)

Let $\Gamma < G$ be any **higher rank lattice**. *Then any normal subgroup $N \lhd \Gamma$ is either finite or has finite index.*

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In this talk, we present a new framework to study **higher rank lattices** using operator algebras.

Main Problem

Given a **higher rank lattice** $\Gamma < G$, we want to understand:

1. **IRS**[a] and **URS**[b] of Γ
2. Structure of group C^*-algebras $C^*_\pi(\Gamma)$ where $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$
3. Dynamical properties of the affine action $\Gamma \curvearrowright \mathrm{PD}(\Gamma)$

[a] A **IRS** is a Γ-invariant Borel probability measure on $\mathrm{Sub}(\Gamma)$.
[b] A **URS** is a nonempty minimal Γ-invariant closed subset of $\mathrm{Sub}(\Gamma)$.

Bibliography

The present talk is based on two joint works:

[BH19] R. Boutonnet, C. Houdayer, *Stationary characters on lattices of semisimple Lie groups.* Publications mathématiques de l'IHÉS, to appear. arXiv:1908.07812

[BBHP20] U. Bader, R. Boutonnet, C. Houdayer, J. Peterson, *Charmenability of arithmetic groups of product type.* arXiv:2009.09952

Main results

The dynamical system $\Gamma \curvearrowright \mathrm{PD}(\Gamma)$

For any countable discrete group Γ, set

$$\mathrm{PD}(\Gamma) \doteq \{\varphi : \Gamma \to \mathbb{C} \mid \text{normalized positive definite function}\}$$

Then $\mathrm{PD}(\Gamma) \subset \ell^\infty(\Gamma)$ is a weak-$*$ compact convex set.

We consider the affine action $\Gamma \curvearrowright \mathrm{PD}(\Gamma)$ given by conjugation

$$\forall \gamma \in \Gamma, \quad \gamma\varphi \doteq \varphi \circ \mathrm{Ad}(\gamma^{-1})$$

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Definition

A **character** $\varphi \in \mathrm{PD}(\Gamma)$ is a fixed point for $\Gamma \curvearrowright \mathrm{PD}(\Gamma)$.

Examples

- For any tracial von Neumann algebra (M, τ) and any unirep $\pi : \Gamma \to \mathcal{U}(M)$, $\varphi \doteq \tau \circ \pi$ is a character.
- $\varphi \doteq \delta_e$ is called the **regular character**: $\pi_\varphi = \lambda$.

Charmenability of higher rank lattices

Theorem (BH19, BBHP20)

Let $\Gamma < G$ be any **higher rank lattice***. Then*

1. *Any nonempty Γ-invariant weak-$*$ compact convex subset $\mathcal{C} \subset \mathrm{PD}(\Gamma)$ contains a character.*
2. *Any extremal character φ is either supported on $\mathcal{Z}(\Gamma)$ or the corresponding GNS tracial factor $\pi_\varphi(\Gamma)''$ is amenable.*

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2. *Any extremal character φ is either supported on $\mathcal{Z}(\Gamma)$ or the corresponding GNS tracial factor $\pi_\varphi(\Gamma)''$ is amenable.*

When G has property (T) (e.g. $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$), we can strengthen the above second item as follows:

2. Any extremal character φ is either supported on $\mathcal{Z}(\Gamma)$ or the corresponding GNS tracial factor $\pi_\varphi(\Gamma)''$ is finite dimensional.

Our theorem strengthens celebrated results by Margulis (1978), Stuck–Zimmer (1992), Bekka (2006), Peterson (2014).

Structure theorem for group C^*-algebras $C^*_\pi(\Gamma)$

When $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ is a unirep, we may regard

$$\mathfrak{S}(C^*_\pi(\Gamma)) \hookrightarrow \mathrm{PD}(\Gamma) : \psi \mapsto \psi \circ \pi$$

as a Γ-invariant weak-$*$ compact convex subset. We obtain:

Theorem (BH19, BBHP20)

Let $\Gamma < G$ be any **higher rank lattice**. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be any unirep. Then $C^*_\pi(\Gamma)$ admits a trace.*

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Assume that G has trivial center. If π is not amenable[a], then $\lambda \prec \pi$, that is, there is a $*$-homomorphism $\Theta : C^*_\pi(\Gamma) \to C^*_\lambda(\Gamma)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Gamma$. Moreover:

1. *$\tau_\Gamma \circ \Theta$ is the unique trace on $C^*_\pi(\Gamma)$.*
2. *$\ker(\Theta)$ is the unique maximal proper ideal of $C^*_\pi(\Gamma)$.*

[a] π is not amenable if and only if $1_\Gamma \not\prec \pi \otimes \overline{\pi}$.

When G has property (T), π weakly mixing $\Rightarrow \lambda \prec \pi$ & Items 1, 2.

Structure theorem for topological dynamics

Theorem (BH19, BBHP20)

Let $\Gamma < G$ be any **higher rank lattice**. *Assume that G has trivial center. Let $\Gamma \curvearrowright X$ be any minimal action on a compact space.*

Then at least one of the following assertions holds:

- *There exists a Γ-invariant Borel probability measure on X.*
- *The action $\Gamma \curvearrowright X$ is topologically free.*

If G has property (T), then either X is finite or $\Gamma \curvearrowright X$ is top free.

In that case, any **URS** of Γ is finite (Glasner–Weiss' problem 2014).

About the overall strategy

Our main results are consequences of a **dynamical dichotomy** for faithful normal ucp Γ-maps $\theta : M \to \mathrm{L}^\infty(G/P)$ defined on ergodic Γ-von Neumann algebras, where $\Gamma < G$ is a higher rank lattice.

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In [BH19], we treat the case when G is simple with real rank ≥ 2 (e.g. $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$). In that case, we prove a much stronger result: the **noncommutative Nevo–Zimmer theorem**. This method cannot work if G has a rank 1 factor such as $\mathrm{SL}_2(\mathbb{R})$.

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In [BBHP20], we treat the case when $G = G_1 \times G_2$ is a product group (e.g. $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$). However, the method we develop in [BBHP20] cannot work if G is simple.

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In that respect, [BH19] and [BBHP20] are complementary.

A dynamical dichotomy for boundary structures

Structure theory of G/P

Let G be a real connected semisimple Lie group with finite center, no compact factor. Choose $K < G$ a maximal compact subgroup and $P < G$ a minimal parabolic subgroup so that $G = KP$.

Example

If $G = \mathrm{SL}_n(\mathbb{R})$, take $K = \mathrm{SO}_n(\mathbb{R})$ and $P < G$ the subgroup of upper triangular matrices.

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Denote by $\nu_P \in \mathrm{Prob}(G/P)$ the unique K-invariant Borel probability measure.

Theorem (Furstenberg 1962)

For every K-invariant admissible Borel probability measure $\mu_G \in \mathrm{Prob}(G)$, $(G/P, \nu_P)$ is the (G, μ_G)- **Poisson boundary**

$$\mathrm{L}^\infty(G/P, \nu_P) \cong \mathrm{Har}^\infty(G, \mu_G)$$

Boundary structures on von Neumann algebras

Let $\Gamma < G$ be any **higher rank lattice**. Let M be any Γ-von Neumann algebra with separable predual.

Definition (Boundary structure)

Let $\theta : M \to \mathrm{L}^\infty(G/P)$ be any faithful normal ucp Γ-map. We then say that θ is a **boundary structure** on M.

We say that θ is **invariant** if $\theta(M) \subset \mathrm{L}^\infty(G/P)^\Gamma = \mathbb{C}1$.

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A Γ-invariant weakly dense unital separable C^*-subalgebra $A \subset M$ is called a **separable model** for $\Gamma \curvearrowright M$.

Then the restriction $\theta|_A : A \to \mathrm{L}^\infty(G/P)$ gives rise to a measurable Γ-map $\beta : G/P \to \mathfrak{S}(A) : b \mapsto \beta_b$ such that

$$\forall a \in A, \quad \theta(a) : G/P \to \mathbb{C} : b \mapsto \beta_b(a)$$

Boundary structures vs. stationary states

Theorem (Furstenberg 1967)

Let $\Gamma < G$ be any lattice in a **real** *connected semisimple Lie group. Then there exists a probability measure $\mu_\Gamma \in \mathrm{Prob}(\Gamma)$ with full support such that $(G/P, \nu_P)$ is the (Γ, μ_Γ)-***Poisson boundary**

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If $\theta : M \to \mathrm{L}^\infty(G/P)$ is a boundary structure, then $\nu_P \circ \theta$ is a faithful normal μ_Γ-stationary state on M.

Conversely, if φ is a faithful normal μ_Γ-stationary state on M, then

$$\theta : M \to \mathrm{Har}^\infty(\Gamma, \mu_\Gamma) : x \mapsto \left(\gamma \mapsto \varphi(\gamma^{-1}x)\right)$$

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Boundary structures generalize stationary states and are useful when dealing with lattices in **semisimple algebraic groups**.

The noncommutative Nevo–Zimmer theorem

Assume that G is a real connected **simple** Lie group with finite center and real rank ≥ 2 (e.g. $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$).

Theorem (BH19)

Let $\Gamma < G$ be any lattice, M any ergodic Γ-von Neumann algebra and $\theta : M \to \mathrm{L}^\infty(G/P)$ any boundary structure.

Then the following dichotomy holds:

- *Either $\theta : M \to \mathrm{L}^\infty(G/P)$ is invariant.*
- *Or there is a proper parabolic subgroup $P < Q < G$ such that $\mathrm{mult}(\theta) \cong \mathrm{L}^\infty(G/Q)$ as Γ-von Neumann algebras.*

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Our theorem extends Nevo–Zimmer's result (2000) in two ways.

Firstly, we deal with arbitrary (noncommutative) von Neumann algebras.

Secondly, we deal with Γ-actions rather than G-actions.

Dynamical dichotomy for boundary structures

We say that $\phi, \psi \in \mathfrak{S}(A)$ are **pairwise singular** and write $\phi \perp \psi$ if there exists a sequence $(a_k)_k$ in A such that $0 \leq a_k \leq 1$ and $\lim_k \phi(a_k) = 1 = \lim_k \psi(1 - a_k)$.

For higher rank lattices $\Gamma < G$ in arbitrary semisimple Lie groups, we prove the following (weaker yet sufficient) dynamical dichotomy.

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Then the following dichotomy holds:

- *Either $\theta : M \to \mathrm{L}^\infty(G/P)$ is invariant.*
- *Or for some (or every) separable model $A \subset M$, we have $\beta_{\gamma b} \perp \beta_b$ for every $\gamma \in \Gamma \setminus \mathscr{Z}(\Gamma)$ and ν_P-a.e. $b \in G/P$.*

S-adic generalizations

The framework we develop in [BBHP20] allows us to treat irreducible lattices $\Gamma < \prod_I \mathbb{G}_i(\ell_i)$ where for every $i \in I$, ℓ_i is a local field and $\mathbb{G}_i$ is a connected semisimple algebraic ℓ_i-group.

Example (Borel–Harish-Chandra)

Let $n \geq 2$, $k \geq 1$ and $S = \{p_1, \dots, p_k\}$ any finite set of primes.

$$\mathrm{SL}_n(\mathbb{Z}_S) < \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{SL}_n(\mathbb{Q}_{p_k})$$

is an irreducible lattice.

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The next theorem gives new examples of lattices with finite **URS**.

Theorem (BBHP20)

Let $n \geq 2$ and $S \subset \mathcal{P}$ a nonempty (possibly infinite) set of primes. Then any **URS** *and any ergodic* **IRS** *of $\mathrm{SL}_n(\mathbb{Z}_S)$ is finite.*