Noncommutative ergodic theory of higher rank lattices

Cyril Houdayer

Université Paris-Saclay & Institut Universitaire de France

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Let $G$ be any **real connected semisimple Lie group** with finite center, no compact factor and real rank $\geq 2$.

Let $\Gamma \vartriangleleft G$ be any **irreducible lattice**, meaning that $\Gamma \vartriangleleft G$ is a discrete subgroup with finite covolume such that $\Gamma \cdot N \vartriangleleft G$ is dense for every noncentral closed normal subgroup $N \vartriangleleft G$.

**Examples (Minkowski, Borel–Harish-Chandra)**

If $G = \text{SL}_n(\mathbb{R})$ for $n \geq 3$, take $\Gamma = \text{SL}_n(\mathbb{Z})$.

If $G = \text{SL}_n(\mathbb{R}) \times \text{SL}_n(\mathbb{R})$ for $n \geq 2$, take $\Gamma = \text{SL}_n(\mathbb{Z}[\sqrt{d}])$ where $d \in \mathbb{N}$ is square free.

In this talk, we simply say that $\Gamma \vartriangleleft G$ is a **higher rank lattice**.
Let $G$ be any \textbf{real connected semisimple Lie group} with finite center, no compact factor and real rank $\geq 2$.

Let $\Gamma < G$ be any \textbf{irreducible lattice}, meaning that $\Gamma < G$ is a discrete subgroup with finite covolume such that $\Gamma \cdot N < G$ is dense for every noncentral closed normal subgroup $N < G$.

\textbf{Examples (Minkowski, Borel–Harish-Chandra)}

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In this talk, we simply say that $\Gamma < G$ is a \textbf{higher rank lattice}. 
Margulis’ Normal Subgroup Theorem (1978)

Let $\Gamma \triangleleft G$ be any higher rank lattice. Then any normal subgroup $N \triangleleft \Gamma$ is either finite or has finite index.
Motivation

Margulis’ Normal Subgroup Theorem (1978)

Let $\Gamma \subset G$ be any higher rank lattice. Then any normal subgroup $N \vartriangleleft \Gamma$ is either finite or has finite index.

In this talk, we present a new framework to study higher rank lattices using operator algebras.

Main Problem

Given a higher rank lattice $\Gamma \subset G$, we want to understand:

1. IRS$^a$ and URS$^b$ of $\Gamma$
2. Structure of group $C^*$-algebras $C^*_\pi(\Gamma)$ where $\pi : \Gamma \to U(H_\pi)$
3. Dynamical properties of the affine action $\Gamma \curvearrowright PD(\Gamma)$

$^a$A IRS is a $\Gamma$-invariant Borel probability measure on $Sub(\Gamma)$.
$^b$A URS is a nonempty minimal $\Gamma$-invariant closed subset of $Sub(\Gamma)$. 
The present talk is based on two joint works:


Main results
The dynamical system $\Gamma \curvearrowright \text{PD}(\Gamma)$

For any countable discrete group $\Gamma$, set

$$\text{PD}(\Gamma) \doteq \{ \varphi : \Gamma \to \mathbb{C} \mid \text{normalized positive definite function} \}$$

Then $\text{PD}(\Gamma) \subset \ell^\infty(\Gamma)$ is a weak-$*$ compact convex set.

We consider the affine action $\Gamma \curvearrowright \text{PD}(\Gamma)$ given by conjugation

$$\forall \gamma \in \Gamma, \quad \gamma \varphi \doteq \varphi \circ \text{Ad}(\gamma^{-1})$$
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**Definition**

A **character** $\varphi \in \text{PD}(\Gamma)$ is a fixed point for $\Gamma \curvearrowright \text{PD}(\Gamma)$.

**Examples**

- For any tracial von Neumann algebra $(M, \tau)$ and any unirep $\pi : \Gamma \to U(M)$, $\varphi \doteq \tau \circ \pi$ is a character.
- $\varphi \doteq \delta_e$ is called the **regular character**: $\pi \varphi = \lambda$. 

Cyril Houdayer (Paris-Saclay & IUF): Noncommutative ergodic theory of higher rank lattices
Theorem (BH19, BBHP20)

Let $\Gamma \triangleleft G$ be any higher rank lattice. Then

1. Any nonempty $\Gamma$-invariant weak-$\ast$ compact convex subset $\mathcal{C} \subset \text{PD}(\Gamma)$ contains a character.

2. Any extremal character $\varphi$ is either supported on $\mathcal{Z}(\Gamma)$ or the corresponding GNS tracial factor $\pi_{\varphi}(\Gamma)''$ is amenable.

When $G$ has property (T) (e.g. $G = \text{SL}_n(\mathbb{R})$ for $n \geq 3$), we can strengthen the above second item as follows:

2. Any extremal character $\varphi$ is either supported on $\mathcal{Z}(\Gamma)$ or the corresponding GNS tracial factor $\pi_{\varphi}(\Gamma)''$ is finite dimensional.
Charmenability of higher rank lattices

Theorem (BH19, BBHP20)

Let $\Gamma < G$ be any higher rank lattice. Then

1. Any nonempty $\Gamma$-invariant weak-$*$ compact convex subset $C \subset PD(\Gamma)$ contains a character.

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Structure theorem for group $C^*$-algebras $C^*_{\pi}(\Gamma)$

When $\pi : \Gamma \to U(\mathcal{H}_\pi)$ is a unirep, we may regard

$$\mathcal{G}(C^*_{\pi}(\Gamma)) \hookrightarrow \text{PD}(\Gamma) : \psi \mapsto \psi \circ \pi$$

as a $\Gamma$-invariant weak-\text{*} compact convex subset. We obtain:

**Theorem (BH19, BBHP20)**

Let $\Gamma < G$ be any **higher rank lattice**. Let $\pi : \Gamma \to U(\mathcal{H}_\pi)$ be any unirep. Then $C^*_{\pi}(\Gamma)$ admits a trace.

1. $\tau_{\Gamma} \circ \Theta$ is the unique trace on $C^*_{\pi}(\Gamma)$.
2. $\ker(\Theta)$ is the unique maximal proper ideal of $C^*_{\pi}(\Gamma)$.

$a_{\pi}$ is not amenable if and only if $1_{\Gamma} \nleq \pi \otimes \pi$.

When $G$ has property (T), $\pi$ weakly mixing $\Rightarrow \lambda \nleq \pi$ & Items 1, 2.
Structure theorem for group C*-algebras $C^*_\pi(\Gamma)$

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Theorem (BH19, BBHP20)

Let $\Gamma < G$ be any higher rank lattice. Let $\pi : \Gamma \to U(\mathcal{H}_\pi)$ be any unirep. Then $C^*_\pi(\Gamma)$ admits a trace.

Assume that $G$ has trivial center. If $\pi$ is not amenable\textsuperscript{a}, then

$\lambda \prec \pi$, that is, there is a $\ast$-homomorphism $\Theta : C^*_\pi(\Gamma) \to C^*_\lambda(\Gamma)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Gamma$. Moreover:

1. $\tau_{\Gamma} \circ \Theta$ is the unique trace on $C^*_\pi(\Gamma)$.
2. $\ker(\Theta)$ is the unique maximal proper ideal of $C^*_\pi(\Gamma)$.

\textsuperscript{a}$\pi$ is not amenable if and only if $1_\Gamma \nprec \pi \otimes \pi$.

When $G$ has property (T), $\pi$ weakly mixing $\Rightarrow \lambda \prec \pi$ & Items 1, 2.
Theorem (BH19, BBHP20)

Let $\Gamma < G$ be any higher rank lattice. Assume that $G$ has trivial center. Let $\Gamma \curvearrowright X$ be any minimal action on a compact space. Then at least one of the following assertions holds:

- There exists a $\Gamma$-invariant Borel probability measure on $X$.
- The action $\Gamma \curvearrowright X$ is topologically free.

If $G$ has property (T), then either $X$ is finite or $\Gamma \curvearrowright X$ is top free. In that case, any URS of $\Gamma$ is finite (Glasner–Weiss’ problem 2014).
About the overall strategy

Our main results are consequences of a dynamical dichotomy for faithful normal ucp $\Gamma$-maps $\theta : M \to L^\infty(G/P)$ defined on ergodic $\Gamma$-von Neumann algebras, where $\Gamma < G$ is a higher rank lattice.
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The proof of the dynamical dichotomy (which is the hard part) uses **von Neumann algebras** theory and depends heavily on whether the ambient connected semisimple Lie group $G$ is simple or not.
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In [BH19], we treat the case when $G$ is simple with real rank $\geq 2$ (e.g. $G = \text{SL}_n(\mathbb{R})$ for $n \geq 3$). In that case, we prove a much stronger result: the noncommutative Nevo–Zimmer theorem. This method cannot work if $G$ has a rank 1 factor such as $\text{SL}_2(\mathbb{R})$.

In [BBHP20], we treat the case when $G = G_1 \times G_2$ is a product group (e.g. $G = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$). However, the method we develop in [BBHP20] cannot work if $G$ is simple. In that respect, [BH19] and [BBHP20] are complementary.
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A dynamical dichotomy for boundary structures
Structure theory of $G/P$

Let $G$ be a real connected semisimple Lie group with finite center, no compact factor. Choose $K < G$ a maximal compact subgroup and $P < G$ a minimal parabolic subgroup so that $G = KP$.

Example

If $G = \text{SL}_n(\mathbb{R})$, take $K = \text{SO}_n(\mathbb{R})$ and $P < G$ the subgroup of upper triangular matrices.
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**Example**

If $G = \text{SL}_n(\mathbb{R})$, take $K = \text{SO}_n(\mathbb{R})$ and $P < G$ the subgroup of upper triangular matrices.

Denote by $\nu_P \in \text{Prob}(G/P)$ the unique $K$-invariant Borel probability measure.

**Theorem (Furstenberg 1962)**

For every $K$-invariant admissible Borel probability measure $\mu_G \in \text{Prob}(G)$, $(G/P, \nu_P)$ is the $(G, \mu_G)$-Poisson boundary

$$L^\infty(G/P, \nu_P) \cong \text{Har}^\infty(G, \mu_G)$$
Let $\Gamma < G$ be any higher rank lattice. Let $M$ be any $\Gamma$-von Neumann algebra with separable predual.

**Definition (Boundary structure)**

Let $\theta : M \to L^\infty(G/P)$ be any faithful normal ucp $\Gamma$-map. We then say that $\theta$ is a **boundary structure** on $M$. We say that $\theta$ is **invariant** if $\theta(M) \subset L^\infty(G/P)^\Gamma = \mathbb{C}1$. 
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A $\Gamma$-invariant weakly dense unital separable C*-subalgebra $A \subset M$ is called a **separable model** for $\Gamma \curvearrowright M$.

Then the restriction $\theta|_A : A \to L^\infty(G/P)$ gives rise to a measurable $\Gamma$-map $\beta : G/P \to \mathcal{G}(A) : b \mapsto \beta_b$ such that

$$\forall a \in A, \quad \theta(a) : G/P \to \mathbb{C} : b \mapsto \beta_b(a)$$
Boundary structures vs. stationary states

Theorem (Furstenberg 1967)

Let \( \Gamma < G \) be any lattice in a **real** connected semisimple Lie group. Then there exists a probability measure \( \mu_\Gamma \in \text{Prob}(\Gamma) \) with full support such that \((G/P, \nu_P)\) is the \((\Gamma, \mu_\Gamma)\)-Poisson boundary

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If $\theta : M \rightarrow L^\infty(G/P)$ is a boundary structure, then $\nu_P \circ \theta$ is a faithful normal $\mu_\Gamma$-stationary state on $M$.

Conversely, if $\varphi$ is a faithful normal $\mu_\Gamma$-stationary state on $M$, then

$$\theta : M \rightarrow \text{Har}^\infty(\Gamma, \mu_\Gamma) : x \mapsto (\gamma \mapsto \varphi(\gamma^{-1}x))$$

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Boundary structures generalize stationary states and are useful when dealing with lattices in semisimple algebraic groups.
Assume that $G$ is a real connected **simple** Lie group with finite center and real rank $\geq 2$ (e.g. $G = \text{SL}_n(\mathbb{R})$ for $n \geq 3$).

**Theorem (BH19)**

Let $\Gamma < G$ be any lattice, $M$ any ergodic $\Gamma$-von Neumann algebra and $\theta : M \to L^\infty(G/P)$ any boundary structure.

Then the following dichotomy holds:

- Either $\theta : M \to L^\infty(G/P)$ is invariant.
- Or there is a proper parabolic subgroup $P < Q < G$ such that $\text{mult}(\theta) \cong L^\infty(G/Q)$ as $\Gamma$-von Neumann algebras.
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Our theorem extends Nevo–Zimmer’s result (2000) in two ways. Firstly, we deal with arbitrary (noncommutative) von Neumann algebras. Secondly, we deal with $\Gamma$-actions rather than $G$-actions.
We say that $\phi, \psi \in \mathcal{S}(A)$ are \textit{pairwise singular} and write $\phi \perp \psi$ if there exists a sequence $(a_k)_k$ in $A$ such that $0 \leq a_k \leq 1$ and $\lim_k \phi(a_k) = 1 = \lim_k \psi(1 - a_k)$.

For higher rank lattices $\Gamma < G$ in arbitrary semisimple Lie groups, we prove the following (weaker yet sufficient) dynamical dichotomy.
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- Or for some (or every) separable model $A \subset M$, we have $\beta \gamma b \perp \beta b$ for every $\gamma \in \Gamma \setminus \mathbb{Z}(\Gamma)$ and $\nu_{P}$-a.e. $b \in G/P$. 

Cyril Houdayer (Paris-Saclay & IUF) Noncommutative ergodic theory of higher rank lattices
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Cyril Houdayer (Paris-Saclay & IUF)  
Noncommutative ergodic theory of higher rank lattices
The framework we develop in [BBHP20] allows us to treat irreducible lattices $\Gamma < \prod_i G_i(\ell_i)$ where for every $i \in I$, $\ell_i$ is a local field and $G_i$ is a connected semisimple algebraic $\ell_i$-group.

**Example (Borel–Harish-Chandra)**

Let $n \geq 2$, $k \geq 1$ and $S = \{p_1, \ldots, p_k\}$ any finite set of primes.

$$\text{SL}_n(\mathbb{Z}_S) < \text{SL}_n(\mathbb{R}) \times \text{SL}_n(\mathbb{Q}_{p_1}) \times \cdots \times \text{SL}_n(\mathbb{Q}_{p_k})$$

is an irreducible lattice.

The next theorem gives new examples of lattices with finite URS.
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The next theorem gives new examples of lattices with finite URS.

Theorem (BBHP20)

Let $n \geq 2$ and $S \subset \mathcal{P}$ a nonempty (possibly infinite) set of primes. Then any URS and any ergodic IRS of $\text{SL}_n(\mathbb{Z}_S)$ is finite.