Excitation spectrum of dilute trapped Bose gases

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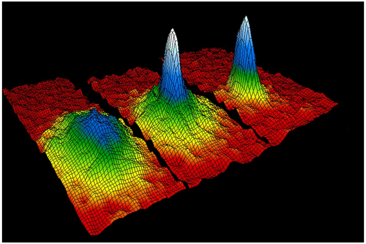
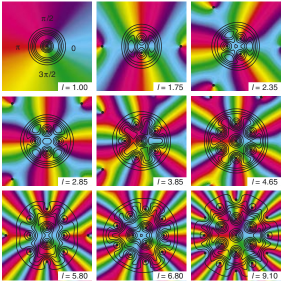
Motivation

- BEC observed for dilute Bose gases
- Quantized vertices observed with rotations $\rightarrow$ related to superfluidity

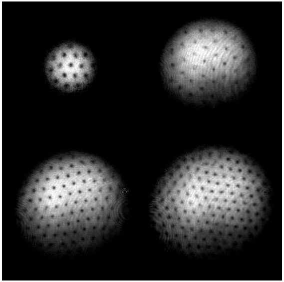
(2001 Nobel Prize in Physics for Cornell–Wieman–Ketterle)



Anderson–Ensher–Matthews–Wieman–Cornell (*Science*, 1995)
Image provided by JILA, University of Colorado, Boulder



Butts–Rokhsar (*Nature*, 1999)



Abo-Shaeer–Raman–Vogels–Ketterle (*Science*, 2001)

Motivation

The Bose–Einstein condensation is closely related to superfluidity

- Allen–Misener & Kapitsa (1938): Superfluid ^{4}He (bosons) at below 2.17 K
- London (1938): Explanation via the Bose–Einstein condensation
- Landau (1941): Theoretical explanation via the excitation spectrum (Nobel Prize in Physics 1962)
- Bogoliubov (1947): Microscopic explanation for Landau's criterion of superfluidity

Today: A mathematical proof of Bogoliubov's theory from first principles

A many–body quantum problem

A system of N bosons in $\mathbb{R}^3$ is described by the Hamiltonian

$$H_N = \sum_{j=1}^{N} \Big(-\Delta_{x_j} + V_{\text{ext}}(x_j) \Big) + \sum_{j<k}^{N} N^2 V(N(x_j - x_k)) \quad \text{on} \quad L^2_s((\mathbb{R}^3)^N)$$

- **Trapping potential:** $V_{\text{ext}} \geq 0$, $V(x) \to \infty$ as $|x| \to \infty$
- **Short–range interaction:** $0 \leq V \in L^1(\mathbb{R}^3)$ radial, compactly supported

  The rescaled interaction potential $N^2 V(Nx)$ has the scattering length $\mathfrak{a}_0/N$ with $\mathfrak{a}_0$ the **scattering length** of V $\to$ dilute gas

H_N is a self-adjoint, positive operator and has compact resolvent

We are interested in the **low-lying eigenvalues**

$$0 < \lambda_1(H_N) < \lambda_2(H_N) \leq \lambda_3(H_N) \leq \dots$$

in the limit $N \to \infty$

Gross–Pitaevskii functional

The leading order is captured by the **Gross–Pitaevskii functional**

$$\mathcal{E}_{\rm GP}(u) = \langle u, (-\Delta + V_{\rm ext})u\rangle + 4\pi\mathfrak{a}_0 \int_{\mathbb{R}^3} |u(x)|^4 {\rm d}x$$

Note that the mean–field approximation suggests to consider Hartree states

$$\Psi(x_1, ..., x_N) \approx u^{\otimes N}(x_1, ..., x_N) = u(x_1)...u(x_N)$$

but it leads to a **wrong** functional

$$\mathcal{E}_{\rm H}(u) = \langle u, (-\Delta + V_{\rm ext})u\rangle + \frac{\hat{V}(0)}{2} \int_{\mathbb{R}^3} |u(x)|^4 {\rm d}x$$

since $\hat{V}(0)$ is just the first Born approximation of

$$8\pi\mathfrak{a}_0 = \inf\left\{\int_{\mathbb{R}^3} (2|\nabla f|^2 + V|f|^2), \quad \lim_{|x|\to\infty} f(x) = 1\right\}$$

The strong correlation leads to a subtle **nonlinear correction** to leading order

Leading order results

Theorem (Lieb–Seiringer–Yngvason 2000)

For every $k \in \mathbb{N}$, when $N \to \infty$,

$$\lambda_k(H_N) = N \inf_{\|u\|_{L^2}=1} \mathcal{E}_{\mathrm{GP}}(u) + o(N)$$

Theorem (Lieb–Seiringer 2002-2006)

For every $k \in \mathbb{N}$, the eigenvector $\Psi_N^{(k)}$ of $\lambda_k(H_N)$ exhibits the **complete condensation**

$$\langle \Psi_N^{(k)}, a^*(\varphi)a(\varphi)\Psi_N^{(k)}\rangle = \left\langle \Psi_N^{(k)}, \sum_{i=1}^{N} (P_\varphi)_{x_i} \Psi_N^{(k)} \right\rangle = N + o(N)$$

with $\varphi > 0$ the unique Gross–Pitaevskii minimizer

The eigenvectors $\Psi_N^{(k)}$ are not close to $\varphi^{\otimes N}$ in norm

The optimal rate of both estimates are $\mathcal{O}(1)$

Bogoliubov's excitations

Taylor expansion of the Gross–Pitaevskii functional with $v \perp \varphi$

$$\mathcal{E}_{\mathrm{GP}}\left(\frac{\varphi + v}{\sqrt{1 + \|v\|_{L^2}^2}}\right) = \mathcal{E}_{\mathrm{GP}}(\varphi) + \frac{1}{2}\left\langle \begin{pmatrix} v \\ v \end{pmatrix}, \mathcal{E}''_{\mathrm{GP}}(\varphi) \begin{pmatrix} v \\ v \end{pmatrix} \right\rangle + o\left(\|v\|_{H^1}^2 \right)$$

The Hessian matrix

$$\mathcal{E}''_{\mathrm{GP}}(\varphi) = \begin{pmatrix} D + 8\pi\mathfrak{a}_0\varphi^2 & 8\pi\mathfrak{a}_0\varphi^2 \\ 8\pi\mathfrak{a}_0\varphi^2 & D + 8\pi\mathfrak{a}_0\varphi^2 \end{pmatrix}$$

with $D = -\Delta + V_{\mathrm{ext}} + 8\pi\mathfrak{a}_0\varphi^2 - \mu$, $D\varphi = 0$, can be diagonalized by a real, symplectic matrix

$$\mathcal{V}^* \mathcal{E}''_{\mathrm{GP}}(\varphi)\mathcal{V} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} \sqrt{1+s^2} & s \\ s & \sqrt{1+s^2} \end{pmatrix}$$

with the one-body **excitation operator** on $\mathfrak{H}_+ = \{\varphi\}^\perp \subset L^2(\mathbb{R}^3)$

$$E = \left(D^{1/2}(D + 16\pi\mathfrak{a}_0\varphi^2)D^{1/2} \right)^{1/2}$$

Main result

Theorem (N.-Triay 2021, arXiv:2106.11949)

For every $k \in \mathbb{N}$, when $N \to \infty$,

$$\lambda_k(H_N) - \lambda_1(H_N) = \sum_{i \geq 1} n_i e_i + \mathcal{O}(N^{-1/12}), \quad n_i \in \{0, 1, 2, ...\}$$

where the **elementary excitations** $e_1 \leq e_2 \leq ...$ are the positive eigenvalues of

$$E = \left(D^{1/2}(D + 16\pi \mathfrak{a}_0 \varphi^2) D^{1/2} \right)^{1/2}$$

with $$D = -\Delta + V_{\text{ext}} + 8\pi \mathfrak{a}_0 \varphi^2 - \mu, \quad D \geq 0, \quad D\varphi = 0$$

Similar results obtained by Brennecke–Schlein–Schraven (to appear)

Heuristically, H_N is unitarily equivalent to a non-interacting operator

$$T^* H_N T - \lambda_1(H_N) \approx \mathrm{d}\Gamma(E) = \bigoplus_{n=0}^{\infty} \left(\sum_{i=1}^{n} E_{x_i} \right)$$

where the right side operator acts on Fock space of excited particles $\mathcal{F}(\{\varphi\}^{\perp})$

Earlier results

- Seiringer (2011) and Grech–Seiringer (2013): in the **mean-field regime** $N^2V(Nx) \mapsto N^{-1}V(x)$, the elementary excitations are eigenvalues of

$$E_H = \left(D_H^{1/2}(D_H + K_H)D_H^{1/2}\right)^{1/2}$$

  with $D_H = -\Delta + V_{\rm ext} + V * \varphi^2 - \mu_H, \quad K_H(x,y) = \varphi(x)V(x-y)\varphi(y)$

  Further extensions by Lewin–N.–Serfaty–Solovej, Dereziński-Napiórkowski, N.–Seiringer, Rougerie–Spehner, Pizzo, Bossmann–Petrat–Seiringer

- Boccato–Brennecke–Cenatiempo–Schlein (2019): the homogeneous gas on $\mathbb{T}^3$ has $\varphi = 1$ and $e_p = \sqrt{|p|^4 + 16\pi\mathfrak{a}_0|p|^2}, p \in 2\pi\mathbb{Z}^3$

  This implies **Landau's criterion for superfluidity**

$$\mathrm{d}\Gamma(E) \geq c_0|\text{total momentum}|, \quad c_0 := \inf_{p\neq 0} \frac{e_p}{|p|} > 0$$

  (a drop with velocity $< c_0$ can move in the ground state without friction)

- N.–Napiórkowski–Ricaud–Triay (2020), Brennecke–Schlein–Schraven (2021): **Optimal rate of BEC** for trapped systems in $\mathbb{R}^3$

Fock space formalism

Consider the bosonic Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2_s((\mathbb{R}^3)^n)$. For $g \in \mathfrak{H}$, define the **creation and annihilation operator** $a^*(g), a(g)$ on $\mathcal{F}$

$$(a^*(g)\Psi)(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} g(x_j)\Psi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n+1})$$

$$(a(g)\Psi)(x_1, \ldots, x_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} \overline{g(x_n)}\Psi(x_1, \ldots, x_n)\mathrm{d}x_n, \quad \forall \Psi \in \mathfrak{H}^n, \forall n$$

The **operator-valued distributions** a^*_x and a_x, with $x \in \mathbb{R}^3$

$$a^*(g) = \int_{\mathbb{R}^d} g(x) a^*_x \mathrm{d}x, \quad a(g) = \int_{\mathbb{R}^d} \overline{g(x)} a_x \mathrm{d}x, \quad \forall g \in \mathfrak{H}$$

satisfy the canonical commutation relations (CCR)

$$[a_x, a_y] = 0, \quad [a^*_x, a^*_y] = 0, \quad [a_x, a^*_y] = \delta(x - y), \quad \forall x, y \in \mathbb{R}^3$$

Then

$$H_N = \int_{\mathbb{R}^3} a^*_x(-\Delta_x + V_{\mathrm{ext}}(x))a_x \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} N^2 V(N(x - y)) a^*_x a^*_y a_x a_y \mathrm{d}x\mathrm{d}y$$

Bogoliubov's approximation (Step 1)

In 1947, Bogoliubov proposed a heuristic approximation for the spectrum of

$$H = \int_{\mathbb{R}^3} a_x^*(-\Delta_x + V_{\rm ext}(x))a_x \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} N^{-1}W(x-y)a_x^* a_y^* a_x a_y \mathrm{d}x\mathrm{d}y$$

Step 1 (c-number substitution): Replace the contribution of the condensate $g \in L^2(\mathbb{R}^3)$, $\|g\|_{L^2} = 1$, by a scalar field

$$a_x \approx N_0^{1/2} g(x) + c_x, \quad a_x^* \approx N_0^{1/2}\overline{g(x)} + c_x^*$$

where $N_0 = \langle a^*(g)a(g)\rangle$ and $c_x = a_x$ restricted to excited Fock space $\mathcal{F}(\{g\}^\perp)$

$$H \approx \int_{\mathbb{R}^3} (N_0^{1/2}\overline{g(x)} + c_x^*)(-\Delta_x + V_{\rm ext}(x))(N_0^{1/2}g(x) + c_x)\mathrm{d}x$$
$$+\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} N^{-1}W(x-y)(N_0^{1/2}\overline{g(x)} + c_x^*)(N_0^{1/2}\overline{g(y)} + c_y^*)\times$$
$$\times(N_0^{1/2}g(x) + c_x)(N_0^{1/2}g(y) + c_y)\mathrm{d}x\mathrm{d}y$$

Bogoliubov's approximation (Step 2)

Step 2 (Quadratic reduction). Use the second order expansion with $c_x \ll N_0^{1/2} \approx N^{1/2}$, namely ignore all terms containing more than 2 c_x or c_x^*

$$H \approx N\mathcal{E}_{\rm H}(g) - \frac{1}{2}\int (W*|g|^2)g^2 + \sqrt{N}\Big[c^*((D_H+\mu_H)g) + \text{h.c.}\Big] + \int c_x^*(D_H)_x c_x \mathrm{d}x$$

$$+\frac{1}{2}\iint W(x-y)\overline{g(x)}g(y)c_x^* c_y \mathrm{d}x\mathrm{d}y + \frac{1}{2}\iint W(x-y)\Big[g(x)g(y)c_x^* c_y^* + \text{h.c.}\Big]\mathrm{d}x\mathrm{d}y$$

with the Hartree functional

$$\mathcal{E}_{\rm H}(g) = \langle g, (-\Delta + V_{\rm ext})g\rangle + \frac{1}{2}\iint W(x-y)|g(x)|^2|g(y)|^2 \mathrm{d}x\mathrm{d}y$$

$$D_H = -\Delta + V_{\rm ext} + W*|g|^2 - \mu_H, \quad \mu_H = \int_{\mathbb{R}^3}\left(|\nabla g|^2 + V_{\rm ext}|g|^2 + |g|^2(W*|g|^2)\right)$$

To describe the low energy spectrum, take g minimizing $\mathcal{E}_{\rm H}$ with $\|g\|_{L^2} = 1$. Then $D_H g = 0$ and the linear terms (with one c_x or c_x^*) disappear

We are left with a quadratic expression in terms of c_x and c_x^*

Bogoliubov's approximation (Step 3)

Step 3 (Symplectic diagonalization). The quadratic Hamiltonian can be diagonalized by a unitary operator T on Fock space $\mathcal{F}(\{g\}^\perp)$ of the form

$$T^* c(v) T = c(\sqrt{1+s^2}v) + c^*(sv), \quad T^* c^*(v) T = c^*(\sqrt{1+s^2}v) + c(sv)$$

$$T^* H T \approx N\mathcal{E}_{\mathrm{H}}(g) - \frac{1}{2}\int (W * |g|^2) g^2 + \frac{1}{2}\mathrm{tr}(E_H - D_H - K_H) + \int c_x^* (E_H)_x c_x \mathrm{d}x$$

where $E_H = (D_H^{1/2}(D_H + 2K_H)D_H^{1/2})^{1/2}$ and $K_H(x,y) = g(x)W(x-y)g(y)$

This approximation is correct in the mean-field regime, W independent of N

However, it is wrong in the dilute regime $N^{-1}W(x) = N^2 V(Nx)$

Hartree functional $\mathcal{E}_{\mathrm{H}}$ has to be replaced by the Gross–Pitaevskii functional $\mathcal{E}_{\mathrm{GP}}$

Bogoliubov's approximation (Step 4)

When $N^{-1}W(x) = N^2 V(Nx)$

$$\mathcal{E}_{\rm H}(g) = \langle g, (-\Delta + V_{\rm ext})g\rangle + \frac{1}{2}\iint N^3 V(N(x-y))|g(x)|^2|g(y)|^2 \mathrm{d}x\mathrm{d}y$$

$$\approx \langle g, (-\Delta + V_{\rm ext})g\rangle + \frac{\hat{V}(0)}{2}\int |g(x)|^4 \mathrm{d}x$$

Step 4 (Landau's correction). For dilute limit, everywhere $\hat{V}(0)$ should be replaced by $8\pi\mathfrak{a}_0$ with $\mathfrak{a}_0$ the scattering length of V

All this gives

$$T^*HT \approx N\mathcal{E}_{\rm GP}(\varphi) - 4\pi\mathfrak{a}_0 \int \varphi^4 + \frac{1}{2}{\rm tr}(E - D - 16\pi\mathfrak{a}_0\varphi^2) + \int c_x^* E_x c_x \mathrm{d}x$$

with the excitation operator of the Gross–Pitaevskii functional

$$E = (D^{1/2}(D + 16\pi\mathfrak{a}_0\varphi^2)D^{1/2})^{1/2}$$

where $$D = -\Delta + V_{\rm ext} + 8\pi\mathfrak{a}_0\varphi^2 - \mu, \quad D \geq 0, \quad D\varphi = 0$$

Proof strategy

Consider

$$H_N = \int_{\mathbb{R}^3} a_x^*(-\Delta_x + V_{\mathrm{ext}}(x))a_x \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} N^2 V(N(x-y))a_x^* a_y^* a_x a_y \mathrm{d}x\mathrm{d}y$$

Bogoliubov's approximation can be made rigorous as

$$T_2^* T_c^* T_1^* U H_N U^* T_1 T_c T_2 \quad \approx \quad \mathrm{const} + \int c_x^* E_x c_x \mathrm{d}x + o(1)_{N\to\infty}$$

with suitable unitary transformations U, T_1, T_c, T_2

We follow the general approach proposed by Boccato, Brennecke, Cenatiempo, Schlein (2017-2020, the homogeneous gas on $\mathbb{T}^3$), but take slightly different transformations to deal with inhomogeneous systems in $\mathbb{R}^3$

Excitation operator U

The operator $U = U(\varphi)$ was introduced by Lewin–N.–Serfaty–Solovej (2015)

Any function $\Psi_N \in \mathfrak{H}^N$ admits a unique decomposition

$$\Psi_N = \varphi^{\otimes N}\xi_0 + \varphi^{\otimes N-1} \otimes_s \xi_1 + \varphi^{\otimes N-2} \otimes_s \xi_2 + \ldots + \xi_N$$

with $\xi_k \in \mathfrak{H}_+^k$ with $\mathfrak{H}_+ = \{\varphi\}^\perp \subset L^2(\mathbb{R}^3) = \mathfrak{H}$

This defines a unitary operator $U : \mathfrak{H}^N \to \mathcal{F}_+^{\leq N} = \mathbb{1}^{\mathcal{N}_+ \leq N}\mathcal{F}(\mathfrak{H}_+)$

$$U : \Psi_N \to (\xi_0, \xi_1, \ldots, \xi_N)$$

This operator acts as

$$U^* a_x U = \sqrt{N - \mathcal{N}_+}\varphi(x) + c_x$$

with c_x the annihilation operator and $\mathcal{N}_+$ the number operator on $\mathcal{F}_+ = \mathcal{F}(\mathfrak{H}_+)$.
Thus U rigorously implements **Bogoliubov's c-number substitution**

We have $UH_NU^* \approx \mathbb{1}^{\mathcal{N}_+\leq N}\mathcal{H}\mathbb{1}^{\mathcal{N}_+\leq N}$ with $\mathcal{H}=\sum_{i=1}^4 \mathcal{L}_i$ acts on $\mathcal{F}_+$

$$\begin{aligned}
\mathcal{L}_0 &= (N-\mathcal{N}_+)\int\left(|\nabla\varphi|^2+V_{\rm ext}|\varphi|^2+\frac{1}{2}(N^3V_N*\varphi^2)\varphi^2\right)\\
&\quad -\frac{1}{2}(\mathcal{N}_++1)\int\left(N^3V_N*\varphi^2\right)\varphi^2+{\rm d}\Gamma(-\Delta+V_{\rm ext}+N^3V_N*\varphi^2),\\
\mathcal{L}_1 &= \sqrt{N}c\left(\left(-\Delta+V_{\rm ext}+N^3V_N*\varphi^2\right)\varphi\right)+{\rm h.c.},\\
\mathcal{L}_2 &= \iint N^3V_N(x-y)\varphi(x)\varphi(y)c_x^*c_y{\rm d}x\,{\rm d}y\\
&\quad +\frac{1}{2}\left(1-\frac{\mathcal{N}_+}{N}-\frac{1}{2N}\right)\iint N^3V_N(x-y)\varphi(x)\varphi(y)c_xc_y{\rm d}x\,{\rm d}y+{\rm h.c.},\\
\mathcal{L}_3 &= \sqrt{(1-\mathcal{N}_+/N)_+}\iint N^{5/2}V_N(x-y)\varphi(x)c_y^*c_xc_y{\rm d}x{\rm d}y+{\rm h.c.},\\
\mathcal{L}_4 &= \iint N^2V_N(x-y)c_x^*c_y^*c_xc_y{\rm d}x{\rm d}y.
\end{aligned}$$

In the Gross–Pitaevskii regime, $\mathcal{L}_4\sim N$ and $\mathcal{L}_3\sim 1$. These terms cannot be ignored but have to be renormalized by two unitary transformations T_2 and T_c, which implement the 2nd & $4^{\rm th}$ steps in Bogoliubov's theory simultaneously

First quadratic transformation T_1

In extract the leading order energy, we have to decode the **correlation structure** of particles. Consider the scattering equation

$$-2\Delta f + Vf = 0 \text{ in } \mathbb{R}^3, \quad \lim_{|x|\to\infty} f(x) = 1.$$

Denote $\omega = 1 - f \sim \mathfrak{a}_0|x|^{-1}$ and the rescaled/truncated functions

$$\omega_{\ell_1,N}(x) = \omega(Nx)\chi(xN/\ell_1), \quad \varepsilon_{\ell_1,N} = 2\Delta(\omega_{\ell_1,N}(x) - \omega(Nx)), \quad \chi \approx \mathbb{1}(|x| \leq 1)$$

We define a unitary (Bogoliubov) transformation T_1 on the Fock space $\mathcal{F}_+$ by

$$T_1^* c(g) T_1 = c(\sqrt{1+s_1^2}g) + c^*(s_1 g), \quad \forall g \in \mathfrak{H}_+$$

with

$$s_1(x,y) = -N\omega_{\ell_1,N}(x-y)\varphi(x)\varphi(y)$$

When $1 \ll \ell_1 \ll N$ we have

$$T_1^* \mathcal{H} T_1 \approx \mathrm{const} + \mathrm{d}\Gamma(D) + \iint N^3 \varepsilon_{\ell_1,N}(x-y)\varphi(x)\varphi(y) c_x^* c_y \mathrm{d}x\mathrm{d}y$$
$$+ \left[\frac{1}{2} \iint N^3 \varepsilon_{\ell_1,N}(x-y)\varphi(x)\varphi(y) c_x^* c_y^* \mathrm{d}x\mathrm{d}y + \mathrm{h.c.}\right] + \mathcal{L}_3 + \mathcal{L}_4$$

Thus T_1 extracts the leading constant of order N and replaces the **short-range potential** $V(N(x-y))$ in $\mathcal{L}_2$ by the **longer-range one** $\varepsilon_{\ell_1,N}(x-y)$. Note that $\varepsilon_{\ell_1,N}$ is supported in $|x| \le \mathcal{O}(\ell_1 N^{-1})$ and

$$\int_{\mathbb{R}^3} N^3 \varepsilon_{\ell_1,N} = 8\pi \mathfrak{a}_0 < \int_{\mathbb{R}^3} V(x)\mathrm{d}x = \int_{\mathbb{R}^3} N^3 V(Nx)\mathrm{d}x$$

The idea of using T_1 goes back to the works of Benedikter–de Oliveira–Schlein (2015) and Brennecke–Schlein (2019) in the context of quantum dynamics where they took $\ell_1 \sim N$ to obtain the optimal rate of BEC

Here we take $\ell_1 \ll N$ such that T_1 is “smaller” and it does not change $\mathcal{L}_3$

Cubic transformation T_c

The cubic term $\mathcal{L}_3$ is still of order 1. To remove it we apply a cubic transformation of the form

$$T_c = e^{S^* - S}, \quad S = \mathbb{1}^{\mathcal{N}_+ \leq M} \iint N^{1/2} \varphi(x) \omega_{\ell_2,N}(x-y) c_x^* c_y^* c_y \mathrm{d}x\mathrm{d}y$$

When $1 \ll \ell_2, M \ll N$, we can use $T_c^* A T_c \approx A + \Big[S - S^*, A\Big]$. In particular,

$$\mathcal{L}_3 + \Big[S - S^*, \mathrm{d}\Gamma(D) + \mathcal{L}_4\Big] \approx 0,$$

and hence

$$T_c^* T_1^* \mathcal{H} T_1 T_c \approx \mathrm{const} + \mathrm{d}\Gamma(D) + \iint N^3 \varepsilon_{\ell_1,N}(x-y) \varphi(x) \varphi(y) c_x^* c_y \mathrm{d}x\mathrm{d}y$$
$$+ \Big[\frac{1}{2} \int N^3 \varepsilon_{\ell_1,N}(x-y) \varphi(x) \varphi(y) c_x^* c_y^* + \mathrm{h.c.}\Big] + \mathcal{L}_4$$

The use of the cubic transformation was introduced by Boccato, Brennecke, Cenatiempo, Schlein (2017-2020), which is crucial to obtain the excitation spectrum. Here our choice of T_c is simpler since we did not change $\mathcal{L}_3$ by T_1

Second quadratic transformation T_2

Finally we diagonalize the quadratic expression

$$\mathrm{d}\Gamma(D) + \iint N^3 \varepsilon_{\ell_1,N}(x-y)\varphi(x)\varphi(y) c_x^* c_y \mathrm{d}x\mathrm{d}y + \Big[\frac{1}{2}\int N^3 \varepsilon_{\ell_1,N}(x-y)\varphi(x)\varphi(y) c_x^* c_y^* + \mathrm{h.c.}\Big]$$

by using a quadratic transformation T_2 which is similar to the one used by Grech–Seiringer (2013) in the mean-field regime. This gives

$$T_2^* T_c^* T_1^* U \mathcal{H} U^* T_1 T_c T_2 \approx \mathrm{const} + \mathrm{d}\Gamma(\widetilde{E})$$

with

$$\widetilde{E} = (D^{1/2}(D + 2N^3 \varepsilon_{\ell_1,N}(x-y)\varphi(x)\varphi(y))D^{1/2})^{1/2}$$

If $1 \ll \ell_1 \ll N$, then $\widetilde{E} \to E = (D^{1/2}(D + 16\pi\mathfrak{a}_0\varphi^2)D^{1/2})^{1/2}$ since

$$N^3 \varepsilon_{\ell_1,N}(x) \approx \left(\int N^3 \varepsilon_{\ell_1,N}\right)\delta_0(x) = 8\pi\mathfrak{a}_0\delta_0$$

Here we see again the advantage of taking $\ell_1 \ll N$, such that $\varepsilon_{\ell_1,N}$ is supported in $|x| \leq \mathcal{O}(\ell_1 N^{-1}) \ll 1$. The proof is complete. □

BEC in the thermodynamic limit: An open problem

Consider N bosons in a large torus $\Omega = [0, L]^3$ described by the Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} W(x_i - x_j)$$

We take **thermodynamic limit** $N \to \infty$, $L \to \infty$, $N/L^3 = \rho > 0$ fixed

Conjecture (BEC in the thermodynamic limit)

If $W \geq 0$, then the ground state Ψ_N of H_N **condensates** on $\varphi = L^{-3/2}\mathbb{1}_\Omega$

$$\frac{\langle \Psi_N, a^*(\varphi)a(\varphi)\Psi_N \rangle}{N} \geq c_0 > 0 \quad \text{independently of } L, N$$

Best known: the **Lee–Huang–Yang formula** (1957), $\mathfrak{a}_0$ = scattering length of W

$$\lim_{\substack{N\to\infty \\ N/L^3=\rho}} \frac{E_N}{N} = 4\pi\mathfrak{a}_0\rho \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}_0^3} + o(\sqrt{\rho})_{\rho\to 0}\right)$$

proved by Dyson (57), Lieb–Yngvason (98), Yau–Yin (08), Fournais–Solovej (19)

Outlook

- In the **Gross–Pitaevskii regime**, the excitation spectrum of an interacting Bose gas of N particles can be effectively described by the spectrum of a one-body operator, as predicted by **Bogoliubov's approximation**
- By suitable transformations, we can renormalize short-range interaction potentials to longer-range ones, thus arriving at the **mean-field regime** where Bogoliubov's diagonalization method applies smoothly
- Can we apply this approach to understand the dilute Bose gas in the **thermodynamic limit**?