Graph limits and Markov spaces

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What is a limit theory?

$G_1, G_2, ..., G_n, ...$ more and more similar. Template for large $G_n$?
Plan

Convergence and limit of finite structures: a bit of history
Left and right convergence, limits
- Dense graph sequences and graphons
- Bounded-degree graph sequences and graphings
What happens inbetween?
- Markov spaces and double measure spaces
Why limits?

Early contructions

- Continuous geometries  von Neumann  1936
- Subgraph counts in the limit Erdős-LL-Spencer  1979
- Continuous partition lattices Björner  1986
- Continuous number fields, matroids,...  Björner-LL  1987
- Convergence of metric spaces  Gromov  1989
Graph Limits

• Scaling limits in statistical physics

• Planar/bounded degree graphs
  Benjamini, Schramm 2001

• Dense graphs
  Borgs, Chayes, LL, T. Sós, Szegedy, Vesztergombi 2003

• First order convergence
  Nesetril, Ossona de Mendez 2010
More limits

• Limits of partially ordered sets  Janson  2011
• Limits of permutations
  Hoppen, Kohayakawa, Moreira, Ráth and Menezes Sampaio  2011
• Limits of functions on Abelian groups
  Szegedy  2012, Green, Tao, Ziegler  2011
• Limits of dense hypergraphs  Elek, Szegedy  2012
„more and more similar”?

Complete graphs

Cycles
„...more and more similar”?
„...more and more similar”?

Penrose tilings
„...more and more similar”?

Erdős-Rényi random graphs $G(n, 1/2)$ ($n \to \infty$)
What is a limit theory?

$G_1, G_2, ..., G_n, ...$ more and more similar. Template for large $G_n$?

- notions of convergence
- construction of limit objects
Left and right convergence

\[ F \rightarrow G \rightarrow H \]

- subgraph counts
- neighborhood statistics
- degree distribution
- eigenvalues
- statistical physics models
- chromatic polynomial
- maximum cut
- regularity partitions
- eigenvectors

Pixel pictures

0 0 1 0 0 1 1 0 0 0 1 0 0 1
0 0 1 0 1 0 1 0 0 0 0 0 1 0
1 1 0 1 0 1 1 1 1 0 1 0 1 1
0 0 1 0 1 0 1 0 1 0 1 1 0 0
0 1 0 1 0 1 1 0 0 0 1 0 0 1
1 0 1 0 1 0 1 1 0 1 1 1 0 1
1 1 1 1 1 1 0 1 0 1 1 1 0 0
0 0 1 0 0 1 1 0 1 0 1 0 1 1
0 0 1 1 0 0 0 1 1 1 0 1 0 0
0 0 0 0 0 1 1 0 1 0 1 0 1 0
1 0 1 1 1 1 1 0 1 0 1 1 1
0 0 0 1 0 1 1 0 1 0 1 0 1 0
0 1 1 0 0 0 1 1 0 1 1 0 1
1 0 1 0 1 1 0 1 0 0 1 0 1 0
Limits of dense graph sequences

A random graph with 100 nodes and 2500 edges

1/2
Limits of dense graph sequences

Randomly grown uniform attachment graph with 200 nodes

$1 - \max(x, y)$
Limits of dense graph sequences

Limit objects: graphons

\[ W : [0,1]^2 \rightarrow [0,1], \]

symmetric, measurable

Extending graph theory to graphons

Connectivity, matchings, spectra, automorphisms, ...
Dense graphs: left-convergence

Density of $F$ in $G = \text{prob that random map } V(F) \to V(G)$ preserves edges: \[ t(F, G) = \frac{\# \text{homomorphisms } F \to G}{|V(G)|^{|V(F)|}} \]

$(G_1, G_2, \ldots)$ is left-convergent, if $(t(F, G_1), t(F, G_2), \ldots)$ is convergent $\forall F$. 
Dense graphs: left-convergence to the limit

Density of graph $F$ in graphon $W$: $t(F, W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j) \, dx$

$G_n \to W$, if $t(F, G_n) \to t(F, W)$ for $\forall F$.

Erdős-Rényi random graph: $G(n, \frac{1}{2}) \to W \equiv \frac{1}{2}$
Dense graphs (main facts)

For every left-convergent graph sequence $(G_n)$ there is a graphon $W$ such that $G_n \rightarrow W$.

For every graphon $W$ there is a graph sequence $(G_n)$ such that $G_n \rightarrow W$.

The limit graphon is essentially unique (up to measure preserving transformations and changes of measure 0).
Dense graphs: right-convergence

\[ F \rightarrow G \rightarrow H \]

\[ Q_q(G) = \{G/P : P \text{ a } q\text{-partition of } V(G)\} \]

\[ G_1, G_2, \ldots \text{ is (left)-convergent iff} \]

\[ \forall q \ Q_q(G_n) \text{ (normalized) is convergent in Hausdorff distance} \]
Local convergence of bounded-degree graphs

\[ \frac{\text{# homomorphisms } F \to G_n}{|V(G_n)|} \text{ converges for all connected } F \]

Limit objects: involution-invariant distributions on countable rooted graphs

Soficity: Aldous-Lyons Conjecture
Gromov Problem: \( \forall \text{ countable group is sofic} \)

Benjamini - Schramm

Local convergence of bounded-degree graphs

Limit object: graphing $\equiv$ Borel graph on $[0,1]$ with bounded degree

with “double counting” condition

$$\int_{A} \deg(x, B) \, dx = \int_{B} \deg(x, A) \, dx$$

Extending graph theory to graphings
Matchings, flows, expansion, edge-coloring,...
Intermediate densities

Current focus of research

Continuum node set

Borgs, Chayes, Cohn, Zhao, Frenkel, Backhausz, Kunszenti-Kovács, LL, Szegedy

random node + random edge $\rightarrow$ symm probability measure on $[0,1]^2$
Markov spaces

Markov space: \((J, \mathcal{A}, \eta)\), where \((J, \mathcal{A})\) is a (Borel) sigma-algebra, \(\eta\) is a (symmetric) probability measure on \((J^2, \mathcal{A}^2)\) with equal marginals;

Stationary distribution: marginals \(\pi(X) = \eta(X \times J) = \eta(J \times X)\)

For every Markov space there is a reversible Markov chain with stationary distribution \(\pi\) and ergodic measure \(\eta\), and vice versa.
Double measure spaces

\((J, \mathcal{A}, \eta, \lambda)\), where \((J, \mathcal{A}, \lambda)\) is a probability space, and \((J, \mathcal{A}, \eta)\) is a Markov space.

\(\pi(X) = \lambda(X)\) generalizes regular graphs

Extending graph theory to Markov spaces
Flows, expansion, spectra, random walks,…
Flow theory: circulation

\[ \sum_{j} f(ij) = \sum_{j} f(ji) \]

flow condition

**circulation:** measure on \([0,1]^2\) with equal marginal

**Hoffman Circulation Theorem:**

For \( \forall \) two measures \( \varphi, \psi \) on \([0,1]^2\),

\( \exists \) circulation \( \alpha \) such that \( \varphi \leq \alpha \leq \psi \),

iff \( \varphi \leq \psi \) and \( \varphi(X \times X^c) \leq \psi(X^c \times X) \)
for every \( X \subseteq [0,1] \).
Convergence to Markov spaces

left-convergence

Lyons
Borgs, Chayes, Cohn, Zhao
Frenkel

right-convergence

Kunészenti-Kovács, LL, Szegedy
Backhausz, Szegedy

How to define subgraph densities in Markov spaces?
What are graph limits good for?

• Existence of optima

• Large deviation theory for random graphs

• Templates for solutions of extremal graph problems (finite forcing)

• Local extremal graph theory
Existence of optima

Minimize $x^3 - 6x$ over $x \geq 0$.

minimum is not attained in rationals

$\Rightarrow$ real numbers are useful
Existence of optima

Minimize density of 4-cycles in a graph with edge-density \( \frac{1}{2} \).

always >1/16, arbitrarily close for random graphs

minimum is not attained among graphs \( \Rightarrow \) graph limits are useful

Minimum is attained for constant \( \frac{1}{2} \) graphon only.
Graphon $W$ is finitely forcible: $\exists F_1, \ldots, F_m, \alpha_1, \ldots, \alpha_m$:

\[
\begin{align*}
t(F_1, W) &= \alpha_1 \\
\vdots \\
t(F_m, W) &= \alpha_m
\end{align*}
\]

$\Rightarrow W$ is determined (up to...)
Finitely forcible graphons

constant $p$ functions \hspace{1cm} Chung-Graham-Wilson 1989

$t(K_2, W) = p$, \hspace{0.5cm} $t(C_4, W) = p^4$

complete bipartite graphs \hspace{1cm} Mantel - Turán

$t(K_2, W) = \frac{1}{2}$, \hspace{0.5cm} $t(K_3, W) = 0$

Finitely forcible graphons \hspace{0.5cm} \approx \hspace{0.5cm} templates for optimal graphs

in extremal graph theory \hspace{1cm} ?
Finitely forcible graphons

stepfunctions

LL-T. Sós 2008

LL-Szegedy 2011
Finitely forcible graphons

Finitely forcible: Baire category I

Not finitely forcible: Baire category II

\[ W(x, y) = \begin{cases} \frac{x + y}{2} \\ xy \end{cases} \]
Finitely forcible graphons

Several conjectures

Extremal graph $\Rightarrow$ Finitely forcible $\Rightarrow$ Nice properties
(polynomial size Szemerédi partitions, ...)

Several conjectures $\times$
Finitely forcible graphons

Kral, Cooper, Glebov, Grzesik, Kaiser, Klimosova, L.M.Lovász, Martins, Noel, Sosnovec 2013-2020

arbitrary graphon  finitely forcible
Thank you for your attention!
Dense graphs (further things to define)

• distance of graphs/graphons in which convergence $\Leftrightarrow$ Cauchy

• metric space of graphons (compact)

• regularity partitions of graphons $\rightarrow$ algorithmic theory of graphons

• spectra of graphons

• extremal graphon theory...
Flow of value $\omega$: measure $\varphi$ on $[0,1]^2$

\[
\varphi^1 - \varphi^2 = \omega (\delta_t - \delta_s)
\]

Max-Flow-Min-Cut etc. generalizes rather straightforwardly.
Decomposition of flows into paths

\[ B = \{(s, x_1, \ldots, x_r, t) : x_i \in [0,1]\} : \text{s - t paths} \]

\[ \tau : \text{measure on } B \]

\[ \hat{\tau}(S) = \int_B |S \cap E(P)| \, d\tau(P) : \text{measure on } [0,1]^2 \]

For every acyclic s-t flow \( \varphi \geq 0 \) there is a \( \tau \) with \( \hat{\tau} = \varphi \).

no circulation \( \alpha \) with \( 0 \leq \alpha \leq \varphi \)