On the Connectivity of Branch Loci of Spaces of Curves

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Applied Combinatorial Geometric Topology
8ECM, June 24, 2021

Joint work with A. Costa and other (important) people
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Given an orientable, closed surface $X$ of genus $g \geq 2$ The equivalence:

$$(X, \mathcal{M}(X), \text{complex atlas}) \ (\mathcal{M}(X) = \langle x, y \rangle, \ p(x, y) = 0, \text{the field of meromorphic functions on } X)$$

$X \cong \frac{\mathbb{H}}{\Delta}$, with $\Delta$ a (cocompact) Fuchsian group

$\Delta$ discrete subgroup of $\text{PSL}(2, \mathbb{R})$

$$(X, \mathcal{M}(X), \text{complex curve}) \ (\mathcal{M}(X) = \mathbb{C}[x, y]/p(x, y), \text{the field of rational functions on } X)$$

$$(Y, \text{dianalytic atlas}) \cong (X/\sigma, \sigma \text{ class of anticonformal involution}) \cong \text{real curve } (Y, \text{birational structure}). \ Y \cong \frac{\mathbb{H}}{\Delta}$, with $\hat{\Delta}$ an NEC group

The ovals of the curve $Y$ are the boundary components of the surface $X/\sigma$, the orientability is the one of $X/\sigma$, the genus (is the genus): topological type $t$
(X, complex atlas) \cong \mathbb{H}/\Delta, \text{ with } \Delta \text{ a (cocompact) Fuchsian group}

Surface Fuchsian Group \( \Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod [a_i, b_i] = 1 \rangle \)

▶ Teichmüller space \( \mathcal{T}_g \), space of geometries on a surface of genus \( g \)

\( \mathcal{T}_g = \{ \sigma : \Gamma_0 \to PSL(2, \mathbb{R}) \mid \sigma \text{ injective, } \sigma(\Gamma_0) \text{ discrete} \} / PSL(2, \mathbb{R}) \)

A Riemann surface with prescribed geometry is given by a marked polygon (and all its conjugate by a hyperbolic transformation) in the hyperbolic plane, or the space of conjugacy classes of Fuchsian groups isomorphic to the abstract group \( \Gamma_0 = \langle a_1, b_1, \ldots, a_g, b_g; a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle \).

▶ Moduli space \( \mathcal{M}_g \), space (orbifold) of conformal structures on a surface of genus \( g \)

▶ Mapping Class Group (Teichmüller Modular Group)

\( \mathcal{M}_g^+ = Diff^+(X) / Diff_0(X) = Out(\Gamma_g) \)

▶ Orbifold Universal Covering \( \mathcal{M}_g = \mathcal{T}_g / \mathcal{M}_g^+ \)

\( \mathcal{B}_g \textbf{ Branch Locus} = \textbf{Singular Locus of } \mathcal{M}_g \textbf{ as orbifold} \) (Not the singular set of \( \mathcal{M}_{2,3} \) as algebraic variety, A. Costa- A. Porto for a proof with Fuchsian groups)
Nielsen Realization Theorem (Abikoff 1980, Macbeath for NEC groups)

\[ B_g = \{ X \in \mathcal{M}_g \mid \text{Aut}(X) \neq 1 \} \]

(\( B_2 \) : surfaces with more automorphisms than the hyperelliptic involution)

\( g = 1 \) Euclidean case: \( \mathcal{T}_1 = \mathbb{H}, \mathcal{M}_1 = \text{PSL}(2, \mathbb{Z}), B_1 = \{ i, e^{i\pi/3} \} \), \( \mathcal{M}_1 \) hyperbolic triangle with a vertex at \( \infty \), the nodal curve \( y^2 = x^3 \).

Considering \((X, \text{dianalytic atlas, top. type } t) \cong \mathbb{H}/\hat{\Delta}, \) with \( \hat{\Delta} \) an NEC group \( \mathcal{T}_t^K \) and \( \mathcal{M}_t^K \) the Teichmüller and moduli space of Klein surfaces of topological type \( t \).

\[ \mathcal{M}_t^K = \mathcal{T}_t^K / M(\hat{\Delta}), \quad M(\hat{\Delta}) = \text{Out}(\hat{\Delta}). \]  
Branch locus \( B_t^K \)

Studies of branch locus and moduli spaces:
For \( g = 1 \) Schwarz
For \( g = 2 \) Bolza (1887, moduli of automorphic functions)
For hyperbolic surfaces Harvey, Natanzon, Macbeath.
Deligne-Mumford Completion (going to $\infty$ in $\mathcal{M}_g$)

Curves whose singularities are ordinary double points (nodes), all of whose irreducible components isomorphic to $\mathbb{P}^1$ (or $\hat{\mathbb{C}}$), meet the other irreducible components in at least 3 nodes: stable curves

$\hat{\mathcal{M}}_g = \mathcal{M}_g \cup \{\text{stable curves}\}$

(deforming by varying the coefficients or roots)

Geometrically: Riemann surfaces with a geodesic multicurve pinched to length 0

(deforming by varying the lengths of a system of curves)

Consider the completion $\hat{\mathcal{B}}_g$ of $\mathcal{B}_g$ in $\hat{\mathcal{M}}_g$
Wish: If $B_g$, $B^K_t$, $\hat{B}_g$ connected one can deform a curve with symmetry to another curve with symmetry along a path of curves, all they with symmetry, maybe pinching some multicurve.

1. The branch loci $B_g$ of moduli spaces of hyperbolic Riemann surfaces are disconnected for all genera with the exception of genera 3, 4, 7, 13, 17, 19 and 59.
   In genus 2 Wiman’s curve (of type I) is isolated.

2. It contains several connected components. E.g. $B_g$ contains isolated strata formed by p-gonal RS for genera a multiple $g$ of $(p-1)/2$, at least $2(p-1)/2$
   Question: How much does the no. of connected comp. grow?

3. Considering RS as Klein surfaces, $B^K_{(g,+),0}$ is connected!
   Bartolini-Costa-I-Porto 2010 (RACSAM)

4. $B^K_{(g,+),k}$ is connected (orientable Klein surfaces) Costa-I-Porto 2015 (Geom. Dedic.)

5. $B^K_{(g,-),0}$ is connected ($g = 4, 5$ Bujalance-Etayo-Martínez-Szpietowski 2014)
   In general? (Costa-I-Porto 2021).
6 Considering $\hat{\mathcal{M}}_g$,

- Question 1: Is $\hat{\mathcal{B}}_g$ connected?
- Question 2: Is the locus of stable $p$-gonal curves connected, $p$ odd prime?

7 The hyperelliptic locus is connected (Seppälä 1982), the $p$-gonal locus is in general disconnected, each connected comp. associated to a partition of $0 \mod p$ (González-Diez 1995, Buser-Silhol-Seppälä 1995)

8 The locus of hyperelliptic non-orientable Klein surface with one boundary component is disconnected. It is connected for the corresponding orientable surfaces.
   Costa-I-Porto 2017 (Inter. J. Math.)

9 The completion of the trigonal locus is connected
   Costa-I-Parlier 2014 (Rev. Mat. Complut.)

10 $\hat{\mathcal{B}}_g$ contains isolated strata of dim.1 for genera $g = p - 1, p \geq 11$. These strata consists of $p$-gonal curves
   Costa-I-Parlier 2014 (Rev. Mat. Complut.)

11 The locus of principally polarized abelian varieties (ppav) admitting involutions is connected
   Reyes-Carocca - Rodríguez 2018
Conformal Geometry and Low Dimensional Manifolds

A conference in Honour of Antonio F. Costa
27 June - 1 July 2022. UNED, Ávila
Fuchsian and NEC Groups

- $\Delta$ (cocompact) discrete subgroup of $PSL(2, \mathbb{R})$
- A (compact) Riemann (surface) orbifold of genus $g \geq 2$ \quad $X = \frac{\mathbb{H}}{\Delta}$
- $\Delta$ has presentation:
  - generators: $x_1, \ldots, x_r, a_1, b_1, \ldots, a_h, b_h$
  - relations: $x_i^{m_i}, i = 1 : r, x_1 \ldots x_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1}$
- $X = \frac{\mathbb{H}}{\Delta}$: orbifold with $r$ cone points and underlying surface of genus $g$
- Algebraic structure of $\Delta$ and geometric structure of $X$ are determined by the signature $s(\Delta) = (h; m_1, \ldots, m_r)$
- NEC group $\Delta$ (hyperbolic silvered 2-orbifolds)
  - extra generators: $e_1, \ldots, e_k, c_{i,j}, 1 \leq i \leq k, 1 \leq j \leq r_i + 1$
  - extra relations: $(c_{i,j-1} c_{i,j})^{n_{i,j}}, j = 1, \ldots, r_i, e_i^{-1} c_{i,r_i} e_i^{-1} c_{i,0}, i = 1, \ldots, k$
  - long relation: either $x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1}$
    or $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2$
  - $s(\Delta) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{1,1}, \ldots, n_{1,r_1}), \ldots, (n_k, 1, \ldots, n_k, r_k)\})$.

Singerman 1970-1974
Fundamental polygon

- Area of $\Delta$: area of a fundamental region $P$
  \[ \mu(\Delta) = 2\pi(2h - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i})) \]

- For NEC group
  \[ \mu(\Delta) = 2\pi(\varepsilon h - 2 + k + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{r_i}(1 - \frac{1}{n_{i,j}})) \]

- $X$ hyperbolic equivalent to $P/\langle$pairing$\rangle$

- Every Riemann/Klein orbifold is diconformally equiv. to a Riemann/Klein surface $X$ (uniformized by a surface group $\Gamma_g$, $\Gamma_{(g,\pm,k)}$) Moore 197X, Bujalance 1982, (Armstrong 1984 for structures associated to more general discontinuos groups)
Automorphisms and Morphisms of RS

$G$ finite group of automorphisms of $X_g = \mathbb{H}/\Gamma$, $\Gamma$ a surface group if there exist $\Delta$ Fuchsian/NEC group and epimorphism $\theta : \Delta \to G$ with $\text{Ker}(\theta) = \Gamma$

$\theta$ is the monodromy of the (regular) covering $f : \mathbb{H}/\Gamma \to \mathbb{H}/\Delta$

$\Delta$: lifting to $\mathbb{H}$ of $G$

An automorphism of $X$ will fix the class of the uniformizing Fuchsian/NEC group
A morphism \( f : X = \mathbb{H}/\Lambda \rightarrow Y = \mathbb{H}/\Delta \), given by the group inclusion \( i : \Lambda \rightarrow \Delta \)
Covering \( f \) determined by monodromy \( \theta : \Delta \rightarrow \Sigma \mid_{\Delta : \Lambda} \), \( \Lambda \mid_{\Delta : \Lambda} = \theta^{-1}(STb(1)) \)
(symbol \( \Longleftrightarrow \) \( \Lambda \)-coset \( \Longleftrightarrow \) sheet for \( f \))

Theorem (Singerman 1971) \( \Lambda \) (and so \( i \)) determined \( \theta \) (and \( \Delta \)): If 
\( s(\Delta) = (h; m_1, \ldots, m_r) \), then 
\( s(\Lambda) = (h'; m_{11}, \ldots, m_{1s_1}, \ldots, m'_{r1}, \ldots, m'_{rs_r}) \) iff 
\( \theta : \Delta \rightarrow \Sigma \mid_{\Delta : \Lambda} \) s.t.

i) Riemann-Hurwitz \( \frac{\mu(\Lambda)}{\mu(\Delta)} = |\Delta : \Lambda| \)

ii) \( \theta(x_i) \) product of \( s_i \) cycles each of length \( \frac{m_i}{m'_{i1}}, \ldots, \frac{m_i}{m'_{is_i}} \)

Analogous result for NEC group & Klein surfaces Singerman 1974, Hoare 1990, Pride 1990

\[ \text{locally a cycle of } \theta(x_i) \]
\[ \text{In case of automorphism groups } G, \theta : \Delta \rightarrow G \leq \mathbb{Z} \]
\[ \theta(x_i) \text{ of order } m_i \]
Example: Surfaces of genus 2 with 8 automorphisms. They admit an action of $D_8$ with monodromy \( \Theta: \Delta(0, 2, 2, 2, 4) \rightarrow D_8 \)

\[
\Theta(x_1) = \alpha = (1, 3, 5, 7)(2, 4, 6, 8), \\
\Theta(x_2) = \delta = (1, 2)(4, 7)(3, 8)(6, 5), \\
\Theta(x_3) = s\alpha = (1, 4)(2, 3)(5, 8)(6, 7)
\]

Of course \( \Theta(x_4) = \alpha^2 = (1, 5)(2, 6)(3, 7)(4, 8) \).

No singular pts for order 4.

For one of order 2:

\( e^1 \), \( e^2 \), \( e^3 \) and \( e^5 \).

The area is $2a^8(1/4) = 4a$, so genus is 2. Area $= 4a(q-1)$. 

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$p$-gonal Riemann Surfaces

- A Riemann surface $X$ is called $p$-gonal if it admits a morphism of degree $p$ on the Riemann sphere.
- $X$ is called cyclic $p$-gonal when $X$ has an automorphism $\varphi$ of order $p$ such that $X/\langle \varphi \rangle = \hat{\mathbb{C}}$.
- Case $p = 2$: $X$ hyperelliptic R.S.
- A Riemann surface $X$ is called elliptic-$p$-gonal if it admits a morphism of degree $p$ on a torus.
- $X$ is called cyclic elliptic-$p$-gonal when the morphism is a regular covering.
- **Severi-Castelnuovo inequality**: A $p$-gonal morphism of $X$ is unique if the genus of $X \geq (p - 1)^2$.
- An elliptic-$p$-gonal morphism of $X$ is unique if the genus of $X \geq 2p + (p - 1)^2$. 
Teichmüller and Moduli Spaces

Δ abstract Fuchsian group \( s(Δ) = (h; m_1, \ldots, m_r) \)

\[ \mathcal{T}_Δ = \{ \sigma : Δ \rightarrow PSL(2, \mathbb{R}) \mid \sigma \text{injective, } \sigma(Δ) \text{ discrete } \} / PSL(2, \mathbb{R}) \]

Teichmüller space \( \mathcal{T}_Δ \) has a complex structure of dim \( 3h - 3 + r \), diffeomorphic to a ball of dim \( 6h - 6 + 2r \).

If \( Λ \) subgroup of \( Δ \) \( (i : Λ \rightarrow Δ) \Rightarrow i_* : \mathcal{T}_Δ \rightarrow \mathcal{T}_Λ \) embedding

\( Γ_g \) surface Fuchsian group \( Γ_g \leq Δ \) \( \mathcal{T}_Δ \subset \mathcal{T}_{Γ_g} = \mathcal{T}_g \)

\( G \) finite group \( \mathcal{T}_g^G = \{ [σ] \in \mathcal{T}_g \mid g[σ] = [σ] \forall g \in G \} \neq \emptyset \)

\( \mathcal{T}_g^G \): surfaces with \( G \) as a group of automorphisms.

Mapping class group \( M^+(Δ) = Out(Δ) = \frac{Diff(\mathbb{H}/Δ)}{Diff_0(\mathbb{H}/Δ)} \)

\( Δ = π_1(\mathbb{H}/Δ) \) as orbifold

\( M^+(Δ) \) acts properly discontinuously on \( \mathcal{T}_Δ \) \( \mathcal{M}_Δ = \mathcal{T}_Δ / M^+(Δ) \)
We can give coordinates to this space by considering decomposition in **pairs of pants**: **Fenchel-Nielsen Coordinates**.

A **pairs of pants** is a surface with boundary obtained by taking two identical copies of a right-angle hexagon and gluing 3 of the sides. A pair of pants is homeomorphic to a sphere with three holes, the boundaries are totally geodesic (any point on the boundary has a neighbourhood isometric to a half-disc). Given three positive real numbers \( l_1, l_2, l_3 \), there is a pair of pants whose boundaries have lengths \( l_1, l_2, l_3 \) respectively.

Any hyperbolic surface \( S_g \) admits a decomposition in \( 2g - 2 \) pairs of pants with \( 3g - 3 \) boundaries (there are many such decompositions).

So we have \( 3g - 3 \) parameters that are the lengths of the boundaries in the pant decompositions \((l_1, l_2, \ldots, l_{3g-3}, \ldots)\). The remaining \( 3g - 3 \) parameters \( \theta_1, \ldots, \theta_{3g-3} \) are the **twist parameters**, each one giving the angle along which two pairs of pants are glued together along the common boundary.

\[
(l_1, l_2, \ldots, l_{3g-3}, \theta_1, \ldots, \theta_{3g-3})
\]

(\textcolor{red}{\text{Teichmüller}}) In fact the map assigning to each class of triples the Fenchel-Nielsen parameters is a homeomorphism \( \mathcal{T}_g \rightarrow \mathbb{R}^{6g-6} \).

This map is not only a homeomorphism but also a conformal map \( \mathcal{T}_g \rightarrow \mathbb{C}^{3g-3} \). (\textcolor{red}{\text{Beltrami, Ahlfors}}).
Surfaces with automorphisms: **Branch Locus**

Consider a marked surface \( \sigma(X) \in \mathcal{T}_g \) and \( \beta \in \mathcal{M}_g^+ \), we have

\[
\mathbb{H}/\Delta_g = X \xrightarrow{\sigma} \sigma(X) \\
\downarrow \quad \beta \ast (X) \xrightarrow{\sigma} \sigma \beta (X)
\]

\( \beta[\sigma] = [\sigma] \Leftrightarrow \gamma \in PSL(2, \mathbb{R}), \quad \sigma(\Gamma_g) = \gamma^{-1} \sigma \beta(\Gamma_g) \gamma \)

\( \gamma \) induces an automorphism of \([\sigma(X)]\)

\[
Stb_{\mathcal{M}_g} [\sigma] = \{ \beta \in \mathcal{M}_g | \beta[\sigma] = [\sigma] \} = Aut([\sigma(X)])
\]

\( G = Aut(X) \) finite, determines a conjugacy class of finite subgroups of \( \mathcal{M}_g \), the **symmetry** of \( X \)

\( X_g, Y_g \) equisymmetric if \( Aut(X_g) \) conjugate to \( Aut(Y_g) \)

(\( Aut(X_g) \): **full automorphism group**)

Singerman's list of non-maximal signatures.
Equisymmetric Stratification

Action: $\theta : \Delta \to Aut(X_g) = G$, $\ker(\theta) = \Gamma_g$

$Aut(X_g) = G$ conjugate $Aut(Y_g)$ iff $w \in Aut(G)$, $h \in Diff^+(X)$

$\epsilon, \epsilon' : G \to Diff^+(X)$, $\epsilon'(g) = h\epsilon w(g)h^{-1}$

Two (surface) monodromies $\theta_1, \theta_2 : \Delta \to G$ topologically equiv. actions of $G$

$\begin{align*}
\Delta & \xrightarrow{\theta_1} G \\
\beta \in Aut(\Delta) & \downarrow \quad \downarrow \quad w \in Aut(G) \\
\Delta & \xrightarrow{\theta_2} G
\end{align*}$

$\theta_1, \theta_2$ equiv under $B(\Delta) \times Aut(G)$, $B(\Delta)$ braid group

Broughton (1990): Equisymmetric Stratification

$\mathcal{M}_g^{G, \theta} = \{X \in \mathcal{M}_g \mid \text{symmetry type of } X \text{ is } G\}$

$\overline{\mathcal{M}}_g^{G, \theta} = \{X \in \mathcal{M}_g \mid \text{symmetry type of } X \text{ contains } G\}$

$\mathcal{M}_g^{G, \theta}$ smooth, connected, locally closed alg. var. of $\mathcal{M}_g$, dense in $\overline{\mathcal{M}}_g^{G, \theta}$

$B_g = \bigcup \overline{\mathcal{M}}_g^{G, \theta}$

Costa-I (2008) $B_g = \bigcup \overline{\mathcal{M}}_g^{Cp, \theta}$ (Cornalba 1987 and 2008)
Connectedness, we are interested in \( Y \in \overline{\mathcal{M}}_{g, \theta_1} \cap \overline{\mathcal{M}}_{g, \theta_2} \)

Finding \( \theta : \Delta \to G = \text{Aut}(Y) \) extends both \( \theta_1 : \Delta_1 \to G_1 \) and \( \theta_2 : \Delta_2 \to G_2 \) with \( \text{Ker}(\theta) = \text{Ker}(\theta_1) = \text{Ker}(\theta_2) = \Gamma_g \)

Geometrically
\[
\begin{array}{c}
\mathbb{H}/\Delta_1 \\
\downarrow \quad f_{\theta_1} \\
\mathbb{H}/\Delta \\
\downarrow \quad f_{\theta} \\
\mathbb{H}/\Delta_2 \\
\downarrow \quad f_{\theta_2} \\
G_1 = C_{p_1} \text{ and } G_2 = C_{p_2}
\end{array}
\]

Corresponding to groups:
\[
\begin{array}{c}
\Delta_1 \\
\downarrow \quad \Delta \\
\downarrow \quad \Delta_2 \\
\downarrow \quad C_{p_1} \\
\downarrow \quad G \\
\downarrow \quad C_{p_2}
\end{array}
\]

We need to look at maximal actions of \( C_p \) for isolated strata
Some Results

- Kulkarni (1991). Existence of isolated points in $B_g$ iff $g = 2$ or $2g+1$ a prime $\geq 11$.
  Isolated points are given by actions $\theta : \Delta(0; p, p, p) \to C_p$, $p = 2g + 1$.
  The actions of $C_7$ in $M_3$ extend to actions of $C_{14}$ or $PSL(2, 7)$.
- Bartolini-I (2009): $M_{C_2, \theta}$ and $M_{C_3, \theta'}$ belong to the same connected component of $B_g$.
  All the closed strata induced by actions of $C_2$ or $C_3$ intersect the closed stratum formed by surfaces $X_g$ admitting an automorphism of order 2 with quotient Riemann surface of genus highest possible: $\frac{g}{2}$ for even $g$ and $\frac{g+1}{2}$ for odd $g$.
- Costa-I (2011): $B_g$ contains isolated strata of dimension 1 iff $g+1$ is a prime $\geq 11$.
  The isolated strata are given by actions:
  $\theta_h : \Delta(0; p, p, p, p) \to C_p : \theta_h(x_1) = a, \theta_h(x_2) = a^i, \theta_h(x_3) = a^j$
  $i \neq 1, p - 1, j \neq 1, p - 1, i, p - i, p - 1 - i - j \neq 1, i, j$.
  These actions are maximal and the strata contain no curve with more symmetry.
- **Branch loci in genera four, seven, thirteen, seventeen, nineteen and fifty-nine are connected.**
  GAP-machinery !!
- Bartolini-Costa-I (2013). These are the only genera with connected branch locus.
Actions given isolated stratum of maximal dimension

- $g = 60$, action $\theta : \Delta(0; 5^{32}) \to C_5$:
  \[
  \theta(x_1) = \cdots = \theta(x_{19}) = \alpha, \quad \theta(x_{20}) = \cdots = \theta(x_{24}) = \alpha^2, \quad \theta(x_{25}) = \alpha^3, \quad \theta(x_{26}) = \cdots = \theta(x_{32}) = \alpha^4.
  \]

- $g = 61$, action $\theta : \Delta(1; 5^{30}) \to C_5$:
  \[
  \theta(a) = \theta(b) = 1, \quad \theta(x_1) = \cdots = \theta(x_{23}) = \alpha, \quad \theta(x_{24}) = \cdots = \theta(x_{28}) = \alpha^2, \quad \theta(x_{29}) = \alpha^3, \quad \theta(x_{30}) = \alpha^4.
  \]

- $g = 63$, action $\theta : \Delta(0; 7^{23}) \to C_7$:
  \[
  \theta(x_1) = \cdots = \theta(x_{14}) = \alpha, \quad \theta(x_{15}) = \cdots = \theta(x_{19}) = \alpha^5, \quad \theta(x_{20}) = \alpha^4, \quad \theta(x_{21}) = \cdots = \theta(x_{23}) = \alpha^2.
  \]

- $g = 67$, action $\theta : \Delta(1; 7^{22}) \to C_7$:
  \[
  \theta(a) = \theta(b) = 1, \quad \theta(x_1) = \cdots = \theta(x_{17}) = \alpha, \quad \theta(x_{18}) = \cdots = \theta(x_{20}) = \alpha^6, \quad \theta(x_{21}) = \alpha^3, \quad \theta(x_{22}) = \alpha^4.
  \]

- $g = 71$, action $\theta : \Delta(2; 7^{21}) \to C_7$:
  \[
  \theta(a_i) = \theta(b_i) = 1, \quad i = 1, 2, \quad \theta(x_1) = \cdots = \theta(x_{13}) = \alpha, \quad \theta(x_{14}) = \cdots = \theta(x_{16}) = \alpha^2, \quad \theta(x_{17}) = \theta(x_{18}) = \alpha^5, \quad \theta(x_{19}) = \alpha^3, \quad \theta(x_{20}) = \alpha^4, \quad \theta(x_{21}) = \alpha^6.
  \]
Costa-I-Parlier (2015): The completions in the Deligne-Munford compactification $\tilde{B}_g$ of isolated strata of dim 1 given by the monodromies $\theta_h$ are isolated.

$$(\theta_h : \Delta(0; p, p, p, p) \rightarrow C_p : \theta_h(x_1) = a, \theta_h(x_2) = a^i, \theta_h(x_3) = a^j)$$

The limit points in $\tilde{B}_g$ of every such stratum (given by a monodromy $\theta_h$ with quotient the sphere with four branch points of order $p$) is the covering given by $f_{\theta_h}$ of the limit point of pinched spheres with a decomposition in two pairs of pants, each pair of pants has as boundary two branch points and a curve surrounding two branch points. As in the next slide.
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Consider the (hyperbolic) orbifold of genus 0 with two branch points of order $p$ and a cusp. The cyclic $p$-gonal coverings are given by the monodromies

$$\theta : \Delta(0; p, p, \infty) = \langle y_1, y_2 \mid y_1^p = y_2^p \rangle \rightarrow \langle t \rangle \text{ where } \theta(y_1) = t^a, \theta(y_2) = t^b$$

Two such maps $(t^a, t^b)$ and $(t'^a, t'^b)$ induce equivalent surfaces iif there exists a $c$ such that $a' \equiv ca \mod (p), \ b' \equiv cb \mod (p)$. Each equivalence class of monodromies has a representative of type $(1, j)$. Call $P_j$ the covering given by the monodromy of type $(1, j)$. The limit points of each stratum are

$$P_i + P_{-1 - i + 1 \over j}, \ P_j + P_{-1 - i + 1 \over j}, \ P_i + P_{p-1 - i - j}$$

where $2 \leq i \leq {p-1 \over 2}, \ i < j \leq p - 3, \ p - 1 - i - j \notin \{1, i, j, p - 1, -i, -j\}$

The limit points for other strata of $p$-gonal Riemann surfaces with quotient the sphere with four branch points are

$$P_1 + P_{p-3}, \ P_1 + P_1, \ P_{p-1} + P_{p-1}, \ P_{p-1} + P_{p-1}, \ P_i + P_i, \ P_{-i} + P_{-i} \text{ with } 2 \leq i \leq {p-1 \over 2}$$

and $P_1 + P_{p-i-2}, \ P_i + P_{p-i-2}$ where $2 \leq i \leq {p-1 \over 2}$

Using elementary number theory, the limit points $P_i + P_{-1 - i + 1 \over j}, \ P_j + P_{-1 - i + 1 \over j}, \ P_i + P_{p-1 - i - j}$ do no coincide with limit points of other stratum.

Finally these limit points do not admit any other automorphism.
The branch locus $\beta_{(q,-1)}$ consists of $\frac{q+2}{2}$ connected components if $q$ is even and $\frac{q+1}{2}$ connected components if $q$ is odd.

Consider $Y$, a hyperelliptic surface of type $E = (q, -1)$; $\chi(Y) = 1 - \frac{1}{q} \cdot 2 \cdot q - 1$.

A index $z$ subgroup in $\Delta^*$ is $s(\Delta) = (\frac{q}{2}, 2; z; -22, 1\{2,2\})$.

$\text{Aut}(Y) \leq C_2 \times C_2$ (Bujalance–Etayo–Gamboa–Gromadzki, 1990).

Geometrically, we have the configuration for the action of $\text{Aut}(Y)/\Gamma_Y$:

- A hyperelliptic involution $\Theta: \Lambda \to C_2 \times C_2 = \langle a, b \rangle$ and monodromies $\Theta$:
  - $s(\Lambda) = (\frac{q}{2}, 2; z; -22, 1\{2,2\})$.
  - $s(\Theta^r(a)) = (0; \frac{q}{2}; 2; 22)$.

- $\text{Or}(C_i) = a$; $\text{Or}(e) = a/1d$ according to $r$'s parity.
  - $\text{Or}(C_i) = a$; $\text{Or}(C_{i+1}) = 1d$.

After a bijection $\Theta(C_i) = b$; $\text{Or}(C_{i+1}) = ab$.

The actions given by $\Theta$ are maximal. They produce $\frac{q+2}{2}$ connected components for $q$ even and $\frac{q+1}{2}$ connected components for $q$ odd.
\[ B_{k, \text{Hyp}} \text{ is connected.} \]

Consider again a hyper. with top type \( S = (q, 0, i, 1) \), \( \text{Hyp. involution} \)
\[ y = \frac{1}{n} \text{ and } \frac{1}{2} (4) = \frac{1}{n} \text{ with } \gamma(\Delta) = (0; +; \frac{1}{2}; \frac{1}{2}; \frac{3}{2}; \frac{5}{2}; \frac{7}{2}; \frac{9}{2}) \]

(a disc with \( 2q + 1 \) cone pts)

The groups of automorphisms of \( \frac{1}{2} \) can be dihedral or cyclic.

\[ \text{Aut}_n(y) : \]

\[ \text{C}_n : n \text{ a proper divisor of } 2q + 1 \]
\[ s(\Delta) = (0; +; \frac{1}{2}; \frac{5}{2}; \frac{7}{2}; \frac{9}{2}) \]

\[ \text{C}_2 : n \text{ a proper divisor of } 2q \]
\[ s(\Delta) = (0; +; \frac{1}{2}; \frac{5}{2}; \frac{7}{2}; \frac{9}{2}) \]

\[ D_n : n \text{ an even divisor of } 4q + 2 \]
\[ s(\Delta) = (0; +; \frac{1}{2}; \frac{5}{2}; \frac{7}{2}; \frac{9}{2}) \]

\[ B_m x (2) : n \text{ an even divisor of } 4q + 2 \]
\[ s(\Delta) = (0; +; \frac{1}{2}; \frac{5}{2}; \frac{7}{2}; \frac{9}{2}) \]

(\text{Bujalance, Gáyano, Gamboa, Gromadzki, 1990})
Graphically, consider configurations

They connect rotations and anticonformal involution of top. type

The following configurations show actions connecting all the strata induced by anticonformal involutions.
THANK YOU